Rotational Invariance and the Spin-Statistics Theorem

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- **Main Result:** A system of n coupled and indistinguishable states obey the Fermi-Dirac statistic, while Bose-Einstein statistics follows when the coupling is broken.

- **Coupling, what does it mean?:** Rotationally invariant quantum states can only occur in pairs. These pairs will be referred to as isotropically spin-correlated states (ISC) or coupled states.

- Paired states $\longleftrightarrow$ Fermi-Dirac

Independent states $\longleftrightarrow$ Bose-Einstein.

Two examples of ISC states can be immediately given:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|+\rangle + |-\rangle\,|-\rangle)$$

and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle).$$

However, what is not apparent is that these are the only ISC states permitted for a system of n-particles. This is now proven.

In effect, in order to show that such states exist only for n=2, it is sufficient to show that it is impossible to have three such particles. This follows, since the existence of n ISC particles, presupposes the existence of $n-1$ such particles. Moreover, the proof also throws deeper understanding on the interpretation of Bell's inequality.

In particular, following an argument of Wigner,

$$\{(+,+,-),(+,-,-)\} \subset$$
$$\{(+,+,-),(+,-,-),(-,+,-),(+,-,+)\}$$

implies

$$P\{(+,+,-),(+,-,-)\} \leq$$
$$P\{(+,+,-),(+,-,-),(-,+,-),(+,-,+)\}.$$

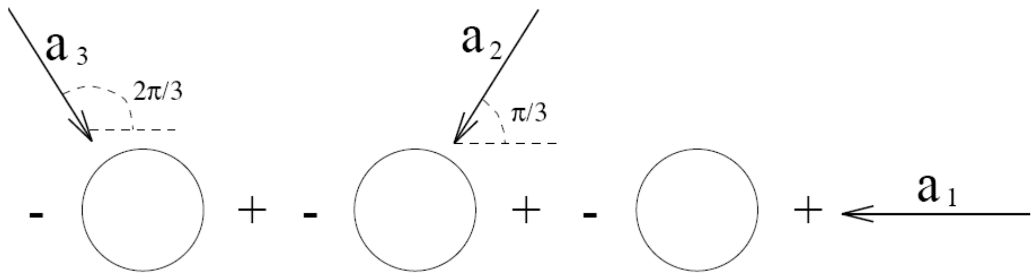

Figure 1: Three isotropically spin-correlated particles.

Therefore,

$$\frac{1}{2}\sin^2\frac{\theta_{ki}}{2} \le \frac{1}{2}\sin^2\frac{\theta_{jk}}{2} + \frac{1}{2}\sin^2\frac{\theta_{ij}}{2}, \qquad (1)$$

which is Bell's inequality. Taking $\theta_{ij} = \theta_{jk} = \frac{\pi}{3}$ and $\theta_{ki} = \frac{2\pi}{3}$ gives $\frac{1}{2} \ge \frac{3}{4}$, a contradiction. In other words, three particles cannot all be in the same spin state with probability 1.

Theorem. The Coupling principle: *Isotropically spin-correlated particles must occur in PAIRS.*

It follows from the coupling principle that multiparticle systems can be divided into two categories, those containing coupled particles and those containing decoupled particles. It now remains to show that a statistical analysis of these two categories, applied to indistinguishable particles, generates the Fermi-Dirac and Bose-Einstein statistics respectively.

Definition. *Two particles whose states are given by* $|\psi(q_1, s_1)\rangle$ *and* $|\psi(q_2, s_2)\rangle$ *respectively are said to be in the same q-orbital when* $q_1 = q_2$.

The following lemma allows us to extend the results for ISC particles defined on the space $H_1 \otimes H_2$ to the larger space $\mathcal{S}_1 \otimes \mathcal{S}_2$, where $\mathcal{S}_i = \mathcal{L}^2(\mathcal{R}^3) \otimes H_i$.

Lemma. *Let*

$$|\psi(\lambda_1, \lambda_2)\rangle = c_1 |\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle + c_2 |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle$$

represent an indistinguishable two particle system defined on the space $\mathcal{S}_1 \otimes \mathcal{S}_2$. If ISC states for a system of two indistinguishable and non-interacting particles occur in the same q-orbital then the system of particles can be represented by the Fermi-Dirac statistics.

The Pauli Exclusion Principle:

Theorem. *A sufficient condition for a state, representing n-cyclically permutable and non-interacting particles, defined on the space* $\mathcal{S}_1 \otimes \ldots \mathcal{S}_n$ *to exhibit Fermi-Dirac statistics is that it contain spin-coupled q-orbitals.*

Remark: A system of n-cyclically permutable particles will be referred to as n indistinguishable particles.

Bose-Einstein Statistics.

Theorem. *Permutable states for a system of n non-interacting particles, defined on the space $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$, obey either the Fermi-Dirac or the Bose-Einstein statistic.*

- Indistinguishable ISC states give Fermi-Dirac Statistics.
- Indistinguishable and statistically independent states give Bose-Einstein statistics.

- Bose-Einstein statistics follow when NO two states are ISC.

- If non-ISC states are statistically independent of each other, then their corresponding spin operators can be represented by the operators $S_i(q_1) \otimes I_2$ and $I_1 \otimes S_i(q_2)$, where I_i represents an identity operator.

- It follows, trivially, that $[S_i(q_1) \otimes I_2, I_1 \otimes S_j(q_2)] = 0$, which means that in the case of Bose-Einstein statistics, spin operators must commute.

- Singlet state $\longleftrightarrow$ Fermi-Dirac statistic.

- Let $s_1(\theta)$ and $s_2(\theta)$ represent spin states, then for an arbitrary angle θ and unit vector $\vec{n}(\theta)$, $\vec{S}_1.\vec{n}(\theta)(s_1(\theta)) = \pm s_1(\theta)$ if and only if $\vec{S}_2.\vec{n}(\theta)(s_2(\theta)) = \mp s_2(\theta)$.

- Identify $s_2(\theta)$ with the orthogonal complement $s_1^-(\theta)$ of $s_1(\theta)$ and put $\vec{S}_1 = \vec{S}_2$.

$$
\begin{aligned}
[S_i(q_1), S_j(q_2)]s &= [S_i(q_1), S_j(q_1)]s \\
&= in\epsilon_{ijk}S_k(q_1)s.
\end{aligned}
$$

- Fermi-Dirac statistics imply spin operators must anticommute and non-local events in the form of spin-singlet states need to be quantized according to the anticommutator rule.

- The above identification is only valid for singlet states, it follows that bosons can never be fermions and fermions can never be bosons.

- Fields quantized by the anticommutator rule *cannot* be quantized with commutators.

Paramagnetism:

- The theory of paramagnetism yields two different equations for the magnetic susceptibility, one given by the classical Langevin (Curie) function which makes no reference to the Pauli exclusion principle and the other which is derived as a direct application of the exclusion principle.

- Our formulation of the exclusion principle gives an equally apt understanding of the phenomenon and would appear to further clarify Pauli's explanation, by focusing on the unique role of the non-spin-singlet states.

- The paired electrons contribute nothing to the magnetic susceptibility. The remaining unpaired electrons act in such a way that there is an excess of electrons in the spin up state over the spin down state, in order to maintain the common electrochemical potential.

Cooper Pairs

- The existence of Cooper pairs as spin-singlet states in the theory of superconductors is another instance of the coupling principle at work.

- The $2n$ superconducting electrons exhibit the statistics of n boson pairs and *not* the usual a_{2n} Fermi-Dirac statistics.

Spin $\frac{3}{2}$ baryons.

• By Theorem 1 it is impossible that a spin $\frac{3}{2}$ baryon be composed of three ISC quarks.

• Moreover, the need of color to explain the structure of Δ^{++} and Ω^{-} particles now becomes both unnecessary and inadequate. The coupling principle forbids the three quarks composing both the Δ^{++} and Ω^{-} particles to exist as ISC particles.

Deuteron Ion

- The deuteron ion is in a spin-triplet state, with spin values X given by +1, 0, -1.

- Conventional quantum mechanics predicts $P(X=+1)=P(X=0)=P(X=-1)=\frac{1}{3}$.

- Model proposed in this paper predicts $P(X=+1)=P(X=-1)=\frac{1}{4}$ and $P(X=0)=\frac{1}{2}$.

- Same results follow using an argument based on Clebsch-Gordan coefficients.

- This should be testable by passing a beam of neutral deuteron atoms (not molecules) through a Stern-Garlach apparatus.