Spin-Statistics

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The paradigm of noncommutative geometry

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Noncommutative geometry
has its roots in Heisenberg' (and Born and Jordan and ...) matrix formulation of quantum mechanics

which resulted in the replacement of the phase space by a 'noncommutative space'

or, more precisely

the replacement of classical observable (usual functions) with quantum ones (infinite dimensional matrices)

Commutativity of algebra of functions on space X
is
localization of points of X

Quantum mechanics phase space noncommutativity:

$$px - xp = \mathrm{i}\hbar \qquad \Rightarrow \qquad \Delta p\, \Delta x \geq \hbar$$

localization of points is ruled out

To incorporate limitations on spatial resolution one makes positions noncommuting as well

$$yx - xy = \mathrm{i}\theta$$

as a consequence

$$\Delta y \, \Delta x \geq \theta$$

Thus spacetime becomes noncommutative

The idea that spacetime geometry may be noncommutative goes back to Schrödinger and Heisenberg

Heisenberg mentioned this possibility in a letter to Rudolph Peierls in the 30s.

Peierls mentions Heisenbergs ideas to Wolfgang Pauli

Pauli explains it to Hartland Snyder

Snyder publishes the first paper on the subject in Physical Review in 1941/42

Model of QHE on deformed (a.k.a. quantum) spheres

Phys motivation:

The Laughlin wave functions for the fractional quantum Hall effect (on the plane) is not translationally invariant.

This problem was overcome by Haldane with a model on a sphere with a magnetic monopole at the origin.

The full Euclidean group of symmetries of the plane is recovered from the rotation group $SO(3)$ of symmetries of the sphere.

One is considering the Hopf fibration of the sphere S^3 over the sphere S^2 with $U(1)$ as gauge (or structure) group

and needs to diagonalize the
Laplacian of S^2 gauged with the monopole connection

The quantum group $\mathrm{SU}_q(2)$

$q = e^{\hbar}$ the deformation parameter

$\mathcal{A} = A(\mathrm{SU}_q(2))$ is the $*$-algebra generated by a and c, and

$$ac = qca, \qquad ac^* = qc^*a, \qquad cc^* = c^*c,$$

$$a^*a + c^*c = aa^* + q^2cc^* = 1$$

these state that the defining matrix is unitary

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

$A(\mathrm{SU}_q(2))$ is a Hopf $*$-algebra (a quantum group)

A U(1) principal bundle.

On $A(\mathsf{SU}_q(2))$ a right coaction of $A(\mathsf{U}(1)) = \mathbb{C} < z, z^{-1} >$:

$$\Delta_R : A(\mathsf{SU}_q(2)) \to A(\mathsf{SU}_q(2)) \otimes A(\mathsf{U}(1))$$

$$\Delta_R \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

the subalgebra of coinvariants

$$A(\mathsf{S}^2_q) := \{p \in A(\mathsf{SU}_q(2)) \ , \ \Delta_R(p) = p \otimes 1\}$$

is Podleś standard sphere. Possible generators:

$$b_- := -q(1+q^2)^{-\frac{1}{2}} \ ac^*, \qquad b_+ := q^2(1+q^2)^{-\frac{1}{2}} \ ca^*$$

$$b_0 := aa^* - (1+q^2)^{-1}$$

The gauged Laplacian

$$\Box_{\nabla} := -\tfrac{1}{2} \star \nabla \star \nabla$$

A remarkable fact:
contrary to the classical limit case, the energies are not symmetric under the exchange $n \leftrightarrow -n$ (this would correspond to inverting the magnetic field axis) , not even when sending $q \leftrightarrow q^{-1}$

$$\lambda_{n,s} = q^{-n-1} \left([s][n+s+1] + \tfrac{1}{2}[n] \right), \qquad \text{for} \quad n \geq 0,$$

$$\lambda_{n,s} = q^{-n-1} \left([s-n][s+1] + \tfrac{1}{2}[n] \right), \qquad \text{for} \quad n \leq 0,$$

Notation

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad \to x \quad \text{as} \quad q \to 1$$

A physics parallel with the quantum Hall effect: the integer s labels Landau levels and the $\phi_{n,s,l}$ are the ('one excitation') Laughlin wave functions with energies $\lambda_{n,s}$.
The lowest Landau, $s = 0$, is $|n|$-degenerate with energy

$$\lambda_{n,0} = \frac{1}{2} q^{-n-1}[|n|]$$

The classical limit. At the value $q = 1$, the energies of the gauged Laplacian become

$$\lambda_{n,s}(q \to 1) = J(J+1) - \frac{1}{4}n^2 = |n|(s + \frac{1}{2}) + s(s+1)$$

and coincide with the energies of the classical gauged Laplacian. They are symmetric under the exchange $n \leftrightarrow -n$ which corresponds to inverting the direction of the magnetic field.

A bit of mathematics

(commutative) Gel'fand-Naimark correspondence:

commutative C^*-algebras $\leftrightarrow$ locally compact Haus. spaces;

points $x \in X$ are characters of $C_0(X)$

$$x(f) := f(x)$$

(or 1-dim irreducible representations)

A (complex, unital) Banach algebra is a complex algebra equipped with a complete normed vector space structure ; the norm and the multiplicative structures are related by the identity

$$\|ab\| \leq \|a\| \|b\|$$

An involutive Banach algebra satisfying the C^*-identity

$$\|a^* a\| = \|a\|^2$$

becomes a C^*-algebra

For $C_0(X)$: $\|f\|_\infty = \sup_{x \in M} |f(x)|$

Pathological spaces: not a good point-set theoretical description

Equivalence relation $\mathcal{R}$ on X;
the quotient $Y = X/\mathcal{R}$ can be bad even for good X

Classically: function on the quotient

$$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \quad ; \quad f \quad \text{is} \quad \mathcal{R} - \text{invariant}\}$$

often not many, only constant functions: $\mathcal{A}(Y) = \mathbb{C}$

NCG approach:

the noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

of functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation (of compact support, of rapid decay,...)

convolution product:

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

involution:

$$f^*(x,y) = \overline{f(y,x)}$$

the quotient $Y = X/\mathcal{R}$ is a noncommutative space

with a noncommutative algebra of functions

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

as good as X to do geometry:

exterior forms, metric, integration,
vector bundles, connections, curvature, ...

with new phenomena coming from noncommutativity

The celebrated irrational rotation algebra a.k.a. the noncommutative torus

$$\mathcal{A}_\theta = C^\infty(\mathbb{T}^2_\theta) \simeq C^\infty(S^1/\theta\mathbb{Z}) \qquad \theta \in \mathbb{R} - \mathbb{Q}$$

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, U_1^m U_2^n \qquad \{a_{mn}\} \in S(\mathbb{Z}^2)$$

$$U_2 U_1 = \mathrm{e}^{2\pi \mathrm{i}\theta}\, U_1 U_2$$

gauge fields on $\mathbb{T}^2_\theta$

A dictionary :

Classical	**Noncommutative**
locally compact space	C^*-algebra
compact space	unital C^*-algebra
vector bundle	finite projective module
smooth manifold	C^*-algebra with ‘smooth’ subalgebra
partial derivative	unbounded derivation
integral	tracial state
spin structure	spectral triple

....

Requirements on noncommutative geometry dictated by physics

The noncommutative geometry of the standard model

Present knowledge of spacetime is described by two theories :

- General Relativity
- The Standard Model of particle physics

Gravity minimally coupled to matter :

$$S = S_{EH} + S_{SM}$$

the gravitational potential $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad\qquad S_{EH}[g_{\mu\nu}] = \frac{1}{G} \int_M R \sqrt{g}\, d^4x$$

the standard model S_{SM} based on the gauge theory of the group

$G = U(1) \times SU(2) \times SU(3)$

The symmetries of $S = S_{EH} + S_{SM}$

$$\mathcal{G} = \mathrm{Map}(M, G) \rtimes \mathrm{Diff}(M)$$

Is there a space so that $\qquad \mathcal{G} = \mathrm{Diff}(X)$?

X can't be a commutative space from the structure of $\mathcal{G}$

it can be a noncommutative space !!

Almost commutative geometry: $X = M \times F$

M Riemannian spin 4-manifold

F finite (i.e. metric 0-dimensional) geometry of KO-dimendion (inner signature) = 6

$\mathcal{A}(X) = C^{\infty}(M) \otimes \mathcal{A}_F$, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})$

$D = \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F$ D_F out of fermion masses

Full standard model minimally coupled to gravity obtained from

$$S(\text{geometry}) = \text{tr}(\chi(D_A/\Lambda)) + \langle J\xi, D_A\xi \rangle$$

$D_A = D + A + \varepsilon' JAJ^{-1}$

J extra structure dictating the KO-dimension (the signature) ; more on this later if times allows

Λ a mass scale, χ is a cut-off function.

Bosonic action counts the eigenvalues of D_A smaller than Λ

The simplest example

$\mathcal{A} = C^\infty(M; M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$

Algebra of $n \times n$ matrices of smooth functions on manifold M

The group $Inn(\mathcal{A})$ of inner automorphisms locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the gauge group $SU(n)$

$$1 \to Inn(A) \to Aut(A) \to Out(A) \to 1$$

becomes identical to

$$1 \to Map(M; G) \to \mathcal{G} \to Diff(M) \to 1$$

(Connes-Chamseddine)

The pure gravity on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$

The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms

What is a metric in spectral geometry

$$d(A;B) = Inf \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$$

Dirac's square root of the Laplacian

$$(\mathcal{A}, \mathcal{H}, D) \qquad ds = D^{-1}$$

$$d(A;B) = Sup_{f \in \mathcal{A}} \{|f(A) - f(B)| \ ; \ ||[D,f]|| \leq 0\}$$

Kinematic relations

- The algebra $\mathcal{A}$ is commutative.

- The commutator $[[D, a], b] = 0$ for $a, b \in A$

this means that the operator D is differential of order one

- It holds that $\sum_\alpha a_0^\alpha \left[[D, a_1^\alpha][D, a_2^\alpha], \ldots, [D, a_n^\alpha]\right] = 1$

for some $a_j^\alpha \in \mathcal{A}$ $[T_1, T_2, \ldots T_n] = \sum_\sigma \epsilon(\sigma) T_{\sigma(1)} T_{\sigma(2)} \cdots T_{\sigma(n)}$

Even case: $1 \to \gamma_5$ (grading)

this means that the determinant of the metric $g^{\mu\nu}$ does not vanish, and in fact more precisely that its square root multiplied by the volume form $\sum_\alpha a_0^\alpha \, da_1^\alpha \, da_2^\alpha \, da_n^\alpha$ gives 1 (or γ_5)

Spectral conditions

- The k-th characteristic value of the resolvent of D goes like

$$\lambda_k \sim k^{-1/n}$$

this condition gives the metric dimension n

- Regularity for the geodesic flow $\exp^{it|D|}$

- Absolute continuity.

The reconstruction

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral geometry fullfing the kinematic relations and the spectral conditions

Assume that the multiplicity of the action of $\mathcal{A}$ in $\mathcal{H}$ is $2^{n/2}$

Then there exists a smooth oriented compact (spin^c) manifold M such that $\mathcal{A} = C^\infty(M)$. Moreover D is a Dirac operator

The Einstein action is obtained by the spectral action

$$\mathrm{tr}(\chi(D/\Lambda))$$

which counting the number of eigenvalues of D of size $< \Lambda$ an expansion in Λ :

$$\mathrm{tr}(\chi(D/\Lambda)) \sim \Lambda^0 \mathrm{vol}(M) + \Lambda^2 \int_M R + O(\Lambda^{-2})$$

To get spin manifolds one needs additional data, a real structure i.e. an antilinear unitary operator J acting in $\mathcal{H}$

It plays the same role and has the same algebraic properties as the charge conjugation operator in physics .

There are additional relations D, J and γ

$$J^2 = \varepsilon(n)1, \qquad JD = \varepsilon'(n)DJ, \qquad J\gamma = \varepsilon''(n)\gamma J$$

$$\begin{array}{rl} \varepsilon(n) = (1, & 1, \ -1, \ -1, \ -1, \ -1, \ 1, \ 1) \\ \varepsilon'(n) = (1, & -1, \ 1, \ 1, \ 1, \ -1, \ 1, \ 1) \\ \varepsilon''(n) = (1, & \ , \ -1, \ \ , \ 1, \ \ , \ -1, \ \ ) \end{array}$$

$n = 0, 1, \ldots, 7$.
In the classical case of spin manifolds, the values of the three signs depend only, upon the value of the dimension n modulo 8

In the classical case of spin manifolds
there is thus a relation between the metric (or spectral) dimension given by the rate of growth of the spectrum of D

and

the integer modulo 8 which appears in the above table

The dimension modulo 8 is called the KO-dimension because of its origin in K-theory ; it is a signature

For more general spaces the two notions of dimension become independent since there are spaces F of metric dimension 0 but of arbitrary KO-dimension

Starting with an ordinary spin geometry M of dimension n and taking the product $M \times F$, one obtains a space whose metric dimension is still n but whose KO-dimension is the sum of n with the KO-dimension of F

It turns out that the Standard Model with neutrino mixing favors the shift of dimension from the 4 of our familiar space-time picture to $10 = 4{+}6 = 2$ modulo 8

Noncommutative spin geometries

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is:

- a unital $*$-algebra $\mathcal{A}$ with a faithful $*$-rep. $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a (separable) Hilbert space,

- a self-adjoint operator D on $\mathcal{H}$ ("the Dirac operator")

(i) $(D + \mathrm{i})^{-1}$ is compact

(ii) $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$

The spectral triple is *graded* (or even) if there exists a $\mathbb{Z}_2$-grading operator γ on $\mathcal{H}$, $\gamma = \gamma^*$, $\gamma^2 = 1$, s.t.

$$\gamma D = -D\gamma, \qquad \pi(a)\gamma = \gamma\pi(a), \qquad a \in \mathcal{A}$$

With $0 < \mu < \infty$, the spectral triple is μ^+-summable (of metric dimension μ) if $(D^2 + 1)^{-1/2}$ is in the Dixmier ideal $\mathcal{L}^{\mu+}(\mathcal{H})$;

$$\text{roughly} \qquad \mu_k(D) \sim k^{1/\mu} \qquad \text{as} \qquad k \to \infty$$

A tracial state (an integral)

$$\int\!\!\!\!\!- \, T = \mathrm{lim}_\omega \frac{1}{\log N} \sum_{k=0}^{N-1} \mu_k(T) \ , \quad \forall \ T \in \mathcal{L}^{\mu+}(\mathcal{H}) \ , \ T \geq 0$$

M a compact Riemannian spin manifold (no boundary)

the canonical spectral triple on M: $(C^\infty(M), \mathcal{H}, D\!\!\!/)$

$\mathcal{H} := L^2(M, \mathcal{S})$ the Hilbert space of spinors;

$D\!\!\!/$ the Dirac operator of the metric of M;

$C^\infty(M)$ act on spinors point-wise

it is m^+-summable (of metric dimension m); $m = dimM$

$(D^2 + 1)^{-1/2} \in \mathcal{L}^{m+}(\mathcal{H})$ is Weyl's formula for the eigen.s of $|D|$:

$$\mu_k(|D|) \sim 2\pi \left(\frac{m}{\Omega_m \mathrm{vol}(M)} \right)^{1/m} k^{1/m} \qquad \text{as} \quad k \to \infty$$

The geodesic distance between any two points on M:

$$d(p,q) = \sup_{f \in C^\infty(M)} \{|f(p) - f(q)| \; ; \; \|[D\!\!\!/, f]\| \leq 1\}$$

The Riemannian measure on M:

$$\int_M d\mu_g \, f \sim ⨍ (f |D\!\!\!/|^{-m})$$

the right hand sides in both furmulæ above
make sense for any spectral triple

in particular p, q are states of the algebra $\mathcal{A}$

Note that $\mathrm{Diff}(M) = \mathrm{Aut}(C^\infty(M))$.

The (Eucledian) Einstein-Hilbert action obtained as:

$$\fint |D\!\!\!/|^{2-m} = -\frac{(m-2)2^{[m/2]}\Omega_m}{12m(2\pi)^m}\int_M R\ d\mu_g$$

An evolution is the Spectral action :

$$S(\text{geometry}) = \text{tr}(\chi(|D\!\!\!/|/\Lambda))$$

with Λ a mass scale, χ is a cut-off function.
The action counts the eigenvalues of D smaller than Λ.

An expansion in Λ :

$$\text{tr}(\chi(D\!\!\!/ /\Lambda)) \sim \Lambda^0 \text{vol}(M) + \Lambda^2 \int_M R + O(\Lambda^{-2})$$

A real structure J of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$;
an antilinear isometry J on $\mathcal{H}$,

$$J^2 = \varepsilon(n)1, \qquad JD = \varepsilon'(n)DJ, \qquad J\gamma = \varepsilon''(n)\gamma J$$

$$\begin{aligned} \varepsilon(n) &= (1, \quad 1, \quad -1, \quad -1, \quad -1, \quad -1, \quad 1, \quad 1) \\ \varepsilon'(n) &= (1, \quad -1, \quad 1, \quad 1, \quad 1, \quad -1, \quad 1, \quad 1) \\ \varepsilon''(n) &= (1, \quad \ , \quad -1, \quad \ , \quad 1, \quad \ , \quad -1, \quad \ ) \end{aligned}$$

$n = 0, 1, \ldots, 7$.

J maps to the commutant

$$[\pi(a), J\pi(b)J^{-1}] = 0, \qquad a, b \in \mathcal{A},$$

the first order condition on D

$$[[D, \pi(a)], J\pi(b)J^{-1}] = 0, \qquad a, b \in \mathcal{A}$$

Standard quantum field theories have many divergence problems; main reason treat particles as point-like.

Problems of standard quantum field theories disappear in non-commutative field theories

vast beautiful new territories out there

thank you !!