On the Geometry of the Berry-Robbins approach to Spin-Statistics

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N.P, M. Paske, A. Reyes, F. Scheck: Annales Mathématiques Blaise Pascal 11, 205 – 220, 2004.

A. Reyes: Thesis University of Mainz, 2006.

1. Motivation

i. Quantum mechanics hides a secret - spin statistic connection may help to find out.

ii. The G-Theory Principle

iii. The approach of Berry Robbins (BR)

Quantum indistinguishability for quantum particles: spin, statistics and the geometric place, Proc R. Soc. London, A, vol 253, 1771-1790, 1997.

M. Berry: talk in Mainz (1999)

- interesting and inspiring, still fascinating
- Proof of the spin statistic theorem was found?
- Objective: find a geometric formulation of this proof:
- Good application of the G-theory principle

G-Theory Principle: explanations

- Formulated 1987 in Mainz – reduction of the Kaluza-Klein Theory of higher dimensions
  - Typical application: A. Heil, N.P., B. Reifenhäuser, F. Scheck, H. Vogel: Anomalies from the Point of View of G-Theory, Journal of Theoretical Physics, vol, 6, no 2, 1989, 237.
  - Generalization: J. Sladkowsi: Generalised G-Theory: International Journal of Theor. Physics, vol. 30, no 4, 517,1991.
- The G-Theory Principle: emphasis on the role of group action: two groups and a manifold
  - The permutation group $G = S_N$ of N particles
  - The SU(2) group for spin
  - The configuration space Q for N identical particles

  We have first to clarify the role of group actions and manifolds.
- The G-Theory Principle

Analogy to the Erlanger Programme of Felix Klein "What is Geometry?"
Here: „What is the physical theory we are studying?"

2. Preparation

N non identical particles moving in IR^3 I

Configuration space : $\tilde{Q}_N = \{(\vec{r}_1, \dots, \vec{r}_N) \in \mathbb{R}^{3N} : \vec{r}_i \neq \vec{r}_j\}$

For N identical particles, the permutation group II

Configuration space: $Q_N = \tilde{Q}/G$ (constrained)

For N = 2, centre of mass and relative coordinates

$$\tilde{Q} \equiv \tilde{Q}_2 \simeq S^2 = \{r\}$$

exchange of particles 1, 2 corresponds to $r \longmapsto -r$

$$G = \mathbb{Z}_2 \qquad Q \equiv Q_2 = \tilde{Q}/\mathbb{Z}_2 = S^2/\mathbb{Z}_2 = \mathbb{R}P^2 = \{[x] = (x, -x)\}$$

$$S^2 \hookrightarrow \mathbb{R}^3 \qquad r \longmapsto x = (x_1, x_2, x_3)$$

two-spin basis $(s = \frac{1}{2})$: $\{|sm_1\rangle \otimes |sm_2\rangle\}_{(m_1, m_2)}$, $m_1, m_2 \in \{\pm\frac{1}{2}\}$

(fixed)

or $\{|jm\rangle\}_{j,m}$ $j \in \{0, 1\}, m \in \{-j, \dots, +j\}$

Standard formalism wave function:

$$\Psi(r) = \sum_{m_1, m_2} f_{m_1 m_2}(r)\, |sm_1\rangle \otimes |sm_2\rangle = \sum_{j,m} f_{jm}(r)\, |jm\rangle$$

Symmetrisation postulate: $f_{1m}(r)$: antisymmetric, $f_{00}(r)$ = symmetric

In the BR approach, the spin basis is moving: $\{|sm_1(r)\rangle \otimes |sm_2(r)\rangle\}$ or $\{|jm(r)\rangle\}$

The wave function: $|\Psi(r)\rangle = \sum_{m_1,m_2} \psi_{m_1,m_2}(r)\, |sm_1(r)\rangle \otimes |sm_2(r)\rangle = \sum_{j,m} \psi_{jm}(r)\, |jm(r)\rangle$

$\psi_{m_1,m_2}(r)$, $\psi_{jm}(r)$: coefficient functions

$|\Psi\rangle$ lives in "two-spin bundle" over S^2 $(\tilde{Q})$

with the single-valuedness constrain: $|\Psi(-r)\rangle = |\Psi(r)\rangle$

We may also think that the wave function lives in a two-spin bundle over $\mathbb{R}P^2(Q)$ without any constrain. We denote this wave function with $|\Phi([X])\rangle$

We expect an equivalent situation:

Wave function $\quad |\Psi(r)\rangle \equiv |\psi_I(r)\rangle \leftrightarrow |\Phi([X])\rangle \equiv |\psi_{II}([X])\rangle$

Two spin vector bundle $\quad \eta = S^2 \times V \leftrightarrow \xi = \mathbb{R}P^2 \tilde{\times} V$

$V \cong \mathbb{C}^4$

$$\xi_I \leftrightarrow \xi_{II}$$

$$\mathrm{I} \leftrightarrow \mathrm{II}$$

What is the precise connection between (I) and (II)?

3. Geometric formulation

Two possible formulations: I and II

In every formulation we consider three objects:

	I	II
The basis manifold	$Q_{\mathrm{I}} \equiv \tilde{Q}$	$Q_{\mathrm{II}} \equiv Q$
The two-spin vector bundle	$\xi_{\mathrm{I}} \equiv \eta$	$\xi_{\mathrm{II}} \equiv \xi$
The wave function	$\|\Psi_{\mathrm{I}}(r)\rangle \equiv \|\Psi(r)\rangle$	$\|\Psi_{\mathrm{II}}([x])\rangle \equiv \|\Phi([x])\rangle$

To find the equivalence between I and II, we start from I and construct II.

In I we have the action of the permutation group G :

$\tilde{Q}$ is a G-space, G is finite, the G-action is free:

$$\rho : \quad G \times \tilde{Q} \rightarrow \tilde{Q}$$
$$\qquad (g, r) \rightarrow \rho_g(r) \equiv g\, r$$

η is also a G-space: G-vector bundle (G-VB)

Definition (G-vector bundle)

$$\eta = (E(\eta), \pi, \tilde{Q})$$

$$\pi : E(\eta) \longrightarrow \tilde{Q} \qquad , \qquad \text{fiber } \pi^{-1}(r) = E(\eta)_r \cong V \cong \mathbb{C}^n$$

$$z \longmapsto \pi(z) = r$$

$$\tau : G \times \eta \longrightarrow \eta$$

$$(g, z) \longmapsto \tau_g(z) \equiv gz$$

i. π is equivariant (consistent with the G-action)

$$\pi \circ \tau_g = \rho_g \circ \pi$$

$$\begin{tikzcd} E(\eta) \arrow[r, "\tau_g"] \arrow[d, "\pi"'] & E(\eta) \arrow[d, "\pi"] \\ \tilde{Q} \arrow[r, "\rho_g"'] & \tilde{Q} \end{tikzcd}$$

ii. $\tau_g : E(\eta)_r \longrightarrow E(\eta)_{gr}$ VR isomorphism

In this category, $\eta_1 \cong_G \eta_2$ G-bundle isomorphism

$|\Psi_I(\cdot)\rangle$: a special section in η : $|\Psi_I(\cdot)\rangle \in \Gamma(\eta)$

$|\Psi_I(\cdot)\rangle$: G-invariant $|\Psi_I(\cdot)\rangle \in \Gamma^{inv}(\eta)$

To obtain II from I by „division“

$Q_{II} = Q_I/G$ projection $q: Q_I \longrightarrow Q_I/G \; (= Q_{II})$

$\xi_{II} = \xi_I/G$ projection $\bar{q}: \xi_I \longrightarrow \xi_I/G \; (= \xi_{II})$

$|\Psi_{II}(\text{I-I})\rangle \in \Gamma(\xi_{II})$ is unconstrained!

assuming we know $|\Psi_{II}\rangle$,

we can take the pull-back of it to obtain the $|\Psi_I\rangle$

To simplify we use also the notation $Q := Q_{II}$, $\xi := \xi_{II}$, $\xi = (E(\xi), \pi, Q)$

Remark: G is not acting on Q and ξ.

The precise connection between I and II is given by:

Theorem (Atiyah):

If G acts freely on Q_I then there is a bijective correspondence between G-bundles (ξ_I) over Q_I and bundles $\xi_{II} = \xi_I/G$ over $Q_{II} = Q_I/G$

So we have $\xi_I = q^* \xi_{II}$ and $|\Psi_I\rangle = q^* |\Psi_{II}\rangle$

The Theorem tells us two things:

1. We start with I : $\xi_{\mathrm{I}} \xrightarrow[\bar{q}]{} \xi_{\mathrm{I}}/G \ (= \xi_{\mathrm{II}})$ and $\xi_{\mathrm{I}} \underset{G}{\cong} q^* \xi_{\mathrm{II}}$

$$Q_{\mathrm{I}} \xrightarrow[q]{} Q_{\mathrm{I}}/G \ (= Q_{\mathrm{II}})$$

2. We start with II $\xi_{\mathrm{II}} = (E(\xi_{\mathrm{II}}), \pi, Q_{\mathrm{II}})$ we obtain $q^*\xi_{\mathrm{II}}$: G-bundle then $q^*\xi_{\mathrm{II}}/G \cong \xi_{\mathrm{II}}$

Pull-back:

$$\begin{tikzcd} & \xi_{\mathrm{II}} \arrow[d, "\pi"] \\ Q_{\mathrm{I}} \arrow[r, "q"'] & Q_{\mathrm{II}} \end{tikzcd} \qquad x \longmapsto q(x) = [x]$$

$$\begin{tikzcd} q^*\xi_{\mathrm{II}} \arrow[d, "\pi"'] & \xi_{\mathrm{II}} \arrow[d, "\pi"] \\ Q_{\mathrm{I}} \arrow[r, "q"'] & Q_{\mathrm{II}} \end{tikzcd}$$

$$E(q^*\xi_{\mathrm{II}})_x := E(\xi_{\mathrm{II}})_{[x]}$$

Example: $N=2$, $\text{spin}=0$, $V=\mathbb{C}$, $G=\mathbb{Z}_2$

Geometric formulation

I

$S^2=\{x=(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2=1\} < \mathbb{R}^3$

II

$\mathbb{R}P^2 = S^2/G = \{[x]\}$

$q: S^2 \longrightarrow \mathbb{R}P^2$

$x \longmapsto [x]=(x,-x) = \mathbb{R}x$

open cover of $\mathbb{R}P^2$: $U_\alpha = \{[x] : x_\alpha \neq 0\}$ $\alpha = 1,2,3$

Three local charts : $\{(U_\alpha, h_\alpha)\}$, $h_1: U_1 \longrightarrow \mathbb{R}^2$

$[x] \longmapsto \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$

I

$\eta = S^2 \times V$

line bundle over S^2

II

$\xi \in \{\xi_+, \xi_-\}$ $\xi_+ = \mathbb{R}P^2 \times V$, $\xi_- = \mathbb{R}P^2 \tilde{\times} V$

line bundle over $\mathbb{R}P^2$

ξ_- : sub-bundle of $\mathbb{R}P^2 \times \mathbb{C}^3$ $\xi_- = (E(\xi), \pi, \mathbb{R}P^2)$

$$E(\xi_-) := \{ ([x], \lambda \Phi(x)) \in \mathbb{R}P^2 \times \mathbb{C}^3 : \lambda \in \mathbb{C}, \; x \in [x] \}$$

$$\phi(x) = (x_1, x_2, x_3) \quad \text{with} \quad |\phi(-x)\rangle = -|\phi(x)\rangle$$

$z \in E(\xi_-)$ $z = ([x], \lambda \phi(x)) = ([x], -\lambda \phi(-x))$

$\pi(z) = [x]$

Local trivialisation:
$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}$$
$$([x], \lambda \phi(x)) \longmapsto ([x], \mathrm{sgn}(x_\alpha)\lambda) \equiv ([x], v)$$

Transition functions: $\Phi_\beta \circ \Phi_\alpha^{-1} : ([x], v) \longmapsto ([x], \mathrm{sgn}(x_\alpha x_\beta) v)$

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$
$$[x] \longmapsto \mathrm{sgn}(x_\alpha x_\beta)$$

Remark: actions of G on $\tilde{Q}(\equiv Q_I)$, $\eta(\equiv \xi_I)$, $C(\tilde{Q})$ and $\Gamma(\eta)$

$$\rho : G \times \tilde{Q} \longrightarrow \tilde{Q}$$

$$\tau : G \times \eta \longrightarrow \eta$$

$$(g, z) \longmapsto \tau_g(z) \equiv \tau_g(x, y) = (\rho_g x, R_g y)$$

$$z \equiv (x, y)$$

$$G \times C(\tilde{Q}) \longrightarrow C(\tilde{Q})$$

$$(g, a) \longmapsto (ga)(x) := a(\rho_{g^{-1}}(x)) \equiv a(g^{-1}x)$$

$$G \times \Gamma(\eta) \longrightarrow \Gamma(\eta)$$

$$(g, s) \longmapsto (gs)(x) := \tau_g(s(g^{-1}x)) = \tau_g(g^{-1}x, |s(g^{-1}x)\rangle)$$

$$s(x) \equiv (x, |s(x)\rangle)$$

if $s \in \Gamma^{inv}(\eta)$: $gs = s$ $\Rightarrow$ $|s(gx)\rangle = R_g |s(x)\rangle$

4. Algebraic formulation

Serre-Swan Theorem: bijective correspondance

$\{$vector-bundles over M$\}$ $\leftrightarrow$ $\{$Finitely generated projective C(M) - moduls$\}$

geometric | *algebraic*

Vector bundle $\xi = (E(\xi), \pi, M)$ $\leftrightarrow$ $\Gamma(\xi)$ sections in ξ

Data: $\{g_{\alpha\beta}\}$ transition functions

$\{\varphi_\alpha\}$ partition of unity of M subordinate to the cover $\{U_\alpha\}$ of M

with $\sum_\alpha |\varphi_\alpha|^2 = 1$,

$\mathcal{A}$ = C(M): functions on M

it exists a free $\mathcal{A}$ - Modul $\mathcal{A}^{n_\xi}$

and a projector $P_\xi : \mathcal{A}^{n_\xi} \to \Gamma(\xi) \hookrightarrow \mathcal{A}^{n_\xi}$

$\Gamma(\xi) = P_\xi(\mathcal{A}^{n_\xi})$

$\mathcal{A}$-valued block matrix : $(P_\xi)_{\alpha\beta} = |\varphi_\alpha| g_{\alpha\beta} |\varphi_\beta|$

Remark: In the algebraic formulation it is easy to show:

$$\Gamma^{inv}(q^* \xi_{II}) \cong \Gamma(\xi_{II})$$

A theorem of A. Reyes shows:

G-actions on the space of functions $C(Q_I)$ determine all information we may have about the I and II point of view.

We put $\tilde{\mathcal{A}} := C(\tilde{Q})$ and $\mathcal{A} := C(Q)$ and $V_R \cong \mathbb{C}^{n_R}$

A representation R of G is given by

R : $G \longrightarrow GL(V_R)$

Irr (G) = $\{$irreducible, unitary representations of G$\}$

G: finite group, Irr(G) = finite set

Decomposition Theorem (A. Rejes, thesis)

The algebra $\tilde{\mathcal{A}}$ of a $\mathcal{A}$ -Modul has the following decomposition :

$$\tilde{\mathcal{A}} = \bigoplus_{R \in Irr(G)} \mathcal{A}_R \quad \text{with} \quad \mathcal{A}_R = \bigoplus_{i=1}^{n_R} \mathcal{A}_{R,i}$$

$\mathcal{A}_{R,i}$ is finitely generated and projective: there exist N_R and P_R so that

$$\mathcal{A}_{R,i} \cong P_R\left(\mathcal{A}^{N_R}\right)$$

Result: G-action on $\tilde{Q}$ (Q_I) and η (ξ_I) give all possible

$$\xi\,(\xi_{II}) \in \{\xi_R\}_{Irr(G)}$$

Example: N = 2, spin = 0, $V = \mathbb{C}$, $G = \mathbb{Z}_2$

$\tilde{\mathcal{A}} = C(\tilde{\mathcal{Q}}) \equiv C(S^2)$, Irr($G$) = $\{R_+, R_-\}$ R_+ : trivial , $R_- : g \mapsto (-1)^g$

$\tilde{\mathcal{A}} = \mathcal{A}_+ \oplus \mathcal{A}_- \equiv C_+(S^2) \oplus C_-(S^2) \equiv$ symmetric $\oplus$ antisymmetric

$\mathcal{A}_+ \equiv C_+(S^2) \cong C(\mathbb{R}P^2) \equiv \mathcal{A}$

$\tilde{\mathcal{A}}, \mathcal{A}_+, \mathcal{A}_-$ are $\mathcal{A}$-moduls

Connection with vector bundles: $\mathcal{A} \equiv \mathcal{A}_+ \leftrightarrow \xi_+$, $\mathcal{A}_- \cong P_- \mathcal{A}^3 \leftrightarrow \xi_-$

$$\mathcal{A}_- \cong P_- \mathcal{A}^3 \cong \Gamma(\xi_-)$$

Construction of P_- :

$$\varphi_\alpha([x]) := \sqrt{x_\alpha^2} \; , \quad \sum_\alpha \varphi_\alpha^2 = \sum_\alpha x_\alpha^2 = x^2 = 1 \; , \quad \alpha \in \{1,2,3\}$$

$$P_- = (P_-)_{\alpha\beta} = g_{\alpha\beta} \varphi_\alpha \varphi_\beta = \mathrm{sign}(x_\alpha x_\beta) |x_\alpha||x_\beta| = x_\alpha x_\beta$$

$$P_- = (x_\alpha x_\beta) = |\phi\rangle\langle\phi| \qquad \phi : \; \phi(x) = (x_1, x_2, x_3)$$

$$P_- \mathcal{A}^3 = \{ \vec{f} \equiv (f_\alpha) : f_\alpha \in C(\mathbb{R}P^2) \} \qquad \Gamma(\xi_-) = \{ s \}$$

$\vec{f} = \vec{f}(s)$: globaly defined $\qquad s = s(\vec{f})$

Results:

1. $$\mathcal{A}_- \cong p_- \mathcal{A}^3 \cong \Gamma(\xi_-)$$

   generators : $\{x_\alpha\} \Leftrightarrow \{\vec{f}_\alpha = x_\alpha \vec{x}\} \Leftrightarrow \{s_\alpha = s(\vec{f}_\alpha)\}$

   elements : $\sum_\alpha a_\alpha([x])x_\alpha \Leftrightarrow \sum_\alpha a_\alpha([x])\vec{f}_\alpha \Leftrightarrow \sum_\alpha a_\alpha([x])\, s_\alpha$ $\qquad a_\alpha \in \mathcal{A} \equiv C(Q)$

2. Connection ∇ on ξ_- $\quad \left(\nabla^{\xi} \Leftrightarrow \nabla^{p_-\mathcal{A}^3} \right)$

   $$\nabla s \Leftrightarrow p_- d\vec{f}$$

3. ∇ is flat, holomology group $\mathbb{Z}_2$

4. ξ_- is $SU(2)$– bundle

5. Parallel transport by means of ∇ is consistent with the $SU(2)$ action!

5. Connection with the BR approach

5.1 Short review of the exchange mechanism (BR 97)

N = 2, spin $s = \frac{1}{2}$

$Q_{I} \equiv \tilde{Q} = S^2 = \{ r \}$ $\qquad Q_{II} \equiv Q = \mathbb{R}P^2 = \{ [x] = (x,-x) : x \in \mathbb{R}^3 \}$

Standard spin basis (fixed) $\qquad |s m_1\rangle \otimes |s m_2\rangle = |m_1, m_2\rangle =: |M\rangle$

Permutation : $\qquad (1, 2) \longrightarrow (2, 1)$

$$|M\rangle = |m_1, m_2\rangle \longmapsto |m_2 m_1\rangle =: |\tilde{M}\rangle$$

To perform the permutation in a continuous way, an exchange group G' is introduced: $\quad G' = SU(2)_{ex}$

$\mathcal{U}$: representation of exchange rotation

Parametrisation $\qquad \mathcal{U} : \quad S^2 \longrightarrow GL(V) \qquad (1)$

$$r \longmapsto \mathcal{U}(r) = e^{-\theta n(r) \cdot E}$$

$$n = e_3 \times r, \quad r = (\theta, \varphi)$$

With $\mathcal{U}(r)$ we define the transported spin basis

$$|M(r)\rangle := \mathcal{U}(r)\, |M\rangle \qquad (2)$$

We obtain the relation

$$|\tilde{M}(-r)\rangle = (-)^{2s}\, |M(r)\rangle \qquad (3)$$

Properties of the transported Spin Basis

1. The map
$$S^2 \longrightarrow \mathbb{C}^{N_s}$$
$$r \longmapsto |M(r)\rangle := \mathcal{U}(r)|M\rangle \qquad (2)$$
smooth for all M.

2. The following „exchange" rule holds:
$$|\bar{M}(-r)\rangle = (-)^{2s}\,|M(r)\rangle \qquad (3)$$

3. The „parallel transport"condition $\langle M'(r(t))|\frac{d}{dt}M(r(t))\rangle = 0$ for all M and M', and for every smooth curve $t \longmapsto r(t)$.

The wave function
$$|\Psi(r)\rangle = \sum_M \Psi_M(r)\,|M(r)\rangle \qquad (4)$$

The single-valuedness requirement:
$$|\Psi(-r)\rangle = |\Psi(r)\rangle \qquad (5)$$

Assuming the above properties, a direct consequence of (5) is the relation
$$\Psi_{\bar{M}}(-r) = (-)^{2s}\,\Psi_M(r) \qquad (6)$$
correct relation between spin and statistics.

5.2 Construction of the „two-spin bundle"

N = 2

U(r) lives in a 10-dimensional space V

A basis of V is an obvious notation

$|e_1\rangle = a_1^+ a_2^+ |0\rangle = |++\rangle$, $|e_2\rangle = b_1^+ b_2^+ |0\rangle = |--\rangle$, $|e_3\rangle = |+-\rangle$, $|e_4\rangle = |-+\rangle$

$|e_5\rangle = a_1^+ b_1^+ |0\rangle$, $|e_6\rangle = a_2^+ b_2^+ |0\rangle$, $|e_7\rangle = (a_1^+)^2 |0\rangle$, $|e_8\rangle$, $|e_9\rangle$, $|e_{10}\rangle$

$$V = \text{span}\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle, |e_5\rangle, |e_6\rangle, \dots |e_{10}\rangle\}$$

The transported spin vectors are maps: S

Instead of $|m_1, m_2\rangle$ we use the total spin basis $|jm\rangle$

For $j=1$: $|m\rangle := |jm\rangle$, $m \in \{-1, 0, +1\}$, for $j=0$ $|00\rangle$, for every m:

$|m\rangle \equiv |m\rangle^{(0)}$: $\{|m\rangle^{(-1)}, |m\rangle^{(0)}, |m\rangle^{(+1)}\}$ ex triplet!

$$j=1 \begin{cases} |-1\rangle : & |-1\rangle^{(-1)} = |e_8\rangle, \quad |-1\rangle^{(0)} = |e_3\rangle, \quad |-1\rangle^{(+1)} = |e_{10}\rangle \\ |0\rangle : & |0\rangle^{(-1)}, \quad |0\rangle^{(0)}, \quad |0\rangle^{(+1)} \\ |+1\rangle : & |+1\rangle^{(-1)} = |e_7\rangle, \quad |+1\rangle^{(0)} = |e_1\rangle, \quad |+\rangle^{(+1)} = |e_9\rangle \end{cases}$$

$V_m = \text{span}(|m\rangle^{(-1)}, |m\rangle^{(0)}, |m\rangle^{(+1)})$ ex triplet space

$\mathcal{U}(r) \in GL(V_m)$

Connection with BR approach

$$\mathcal{U}(r) = \begin{pmatrix} \cos^2\frac{\theta}{2} & -e^{-i\varphi}\frac{\sin\theta}{\sqrt{2}} & e^{-2i\varphi}\sin^2\frac{\theta}{2} \\ e^{i\varphi}\frac{\sin\theta}{\sqrt{2}} & \cos\theta & -e^{i\varphi}\frac{\sin\theta}{\sqrt{2}} \\ e^{2i\varphi}\sin^2\frac{\theta}{2} & e^{i\varphi}\frac{\sin\theta}{\sqrt{2}} & \cos^2\frac{\theta}{2} \end{pmatrix}$$

$$|m r\rangle = \mathcal{U}(r)|m\rangle = \underline{-e^{i\varphi}\frac{\sin\theta}{2}|m\rangle^{(-1)} + \cos\theta\,|m\rangle^{(0)} + e^{i\varphi}\frac{\sin\theta}{\sqrt{2}}|m\rangle^{(+1)}} \quad (*)$$

$$|00r\rangle = |00\rangle$$

Results

1. The exchange rule
   from $(*)$ $\quad \theta = \pi \quad : \quad j=1 \ : \ |jm(-r)\rangle = -|jm(r)\rangle$
   $(j=0 \ : \ |00(-r)\rangle = |00r\rangle)$

2. Line bundles over S^2
   $j=1 \ : \ |jm(r)\rangle$ non vanishing for all r and m
   $r \longmapsto |jm(r)\rangle$ a section in $S^2 \times V_m$
   $\eta_{jm} = \{ |jmr\rangle \mathbb{C} : r \in S^2 \} = S^2 \times \mathbb{C}$

3. The „two spin bundle" over S^2

$$\eta = \bigoplus_{j,m} \eta_{jm} \cong S^2 \times \mathbb{C}^4$$

4. The „two spin bundle" over $\mathbb{R}P^2$ (ξ)

Use $\mathcal{U}(r)$ to obtain the projector $P(r)$

$$P(r) = \mathcal{U}(r)\, P_0\, \mathcal{U}(r)^+$$

$$P(r) = (P_{ij}(r)), \quad P_{ij} \in C(S^2), \ (P_{ij} : \text{even})$$

$$P_{ij}([x]) = P_{ij}(r) \qquad P_{ij} \in C(\mathbb{R}P^2) \equiv \mathcal{A}$$

$$P([x]) = P(r) \qquad P\mathcal{A}^3 \cong \Gamma(\xi_{jm}), \quad \xi_{jm} \cong \mathbb{R}P^2 \tilde{\times} \mathbb{C}$$

$$P\mathcal{A}^3 \cong P_-\mathcal{A}^3 \quad \Rightarrow \quad \xi_{jm} \cong \xi_-$$

$$\xi \cong \xi_{1+1} \oplus \xi_{10} \oplus \xi_{1-1} \oplus \xi_{00} \cong \xi_- \oplus \xi_- \oplus \xi_- \oplus \xi_+$$

5. $|jm(r)\rangle$ parallel with respect to the Grassmann connection

Connection with BR approach

5.3 On the single-valuedness condition

Recalling parts **3.** and **4.**

	I		II
configuration space	$\tilde{Q}\ (S^2)\ , x \equiv r$		$Q\ (\mathbb{R}P^2)\ ,\ [x]$
bundle	$\eta = (E(\eta), \pi, \tilde{Q})$		$\xi = (E(\xi), \pi, Q)$
sections	$\Gamma(\eta)$	$\Gamma^{inv}(\eta) \cong \Gamma(\xi)$	$\Gamma(\xi)$
wave function	$\Psi(x)$	$\Psi^{inv} \equiv \tilde{\Psi} \Leftrightarrow g\tilde{\Psi} = \tilde{\Psi}\ (*)$	$\Psi\ ([x]) = ([x], \lvert\psi([x])\rangle)$

$$\tilde{\Psi}(x) = (x, \lvert\psi(x)\rangle)$$

$$\forall g \in G : \quad \lvert\psi(gx)\rangle = g\lvert\psi(x)\rangle \quad (*) \qquad \lvert\psi([gx])\rangle = \lvert\psi([x])\rangle$$

G-orbit : $Gx_1 = \{x_1, \cdots, x_n\} = \{g_i x_1\}$, $\lvert\psi(x_i)\rangle = g_i \lvert\psi(x_1)\rangle\ (*)$

For a $\mathbb{Z}_2$ – action compatible with Fermi-Bose alternative

$$g\ \lvert jm(r)\rangle = (\mathrm{sign}\, g)^{2s-j+L}\ \lvert jm(gr)\rangle \quad (**)$$

$$\lvert\psi(r)\rangle = \sum_{j,m} \psi_{jm}(r)\ \lvert jm(r)\rangle$$

$$(*), (**) \Rightarrow \quad \psi_{jm}(-r) = (-)^{2s-j+L}\ \psi_{jm}(r) \quad , \quad L = L(\xi) \in \mathbb{N}$$

L : is not fixed

6. Conclusions

- A geometric formulation of the spin-statistic connection in quantum mechanics was presented.
- In this formulation several points were clarified but no solution (proof) was found.
- One condition is missing.