GROUP THEORY OF THE SPIN – STATISTICS CONNECTION

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Based on "The Spin-Statistics Theorem in Arbitrary Dimension"
by L. J. Boya and E. C. G. Sudarshan (Univ. of Texas, Austin)
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I–1, Dimension 3

The spin-statistics theorem (W. Pauli, 1940) justifies to quantize fermions with anticommutators; hence, it leads to the Pauli exclusion principle, a fact discovered much earlier, in the realm of the Old Quantum Theory (January, 1925). Indeed, the Pauli principle is really the differentiating principle in Nature, allowing the formation of structures through the concept of valence, the chemical bond, etc. It is easy to grasp the difference between fermions and boson in ordinary, three-dimensional space: the space symmetry group is SO(3), which is implemented in quantum mechanics by projective representations; these are of two kinds: bona-fide, linear representations of SO(3), that is, integer angular momentum and linear representations of the universal covering group, SU(2), which generates also projective representations of SO(3).

I–2, Dimension 3

As the kernel $SU(2) \longrightarrow SO(3)$ is $\mathbb{Z}_2$, all can be seen from the SU(2) perspective, as it is usually done in quantum mechanical textbooks:

- integer angular momentum corresponds to SO(3) **tensors** D_ℓ
- half-integer ang. momentum corresponds to SO(3) **spinors** D_j

Under a 2π rotation, tensors come back to themselves, of course, whereas spinors acquire a minus sign. So now we can state the spin-statistics connection (in our supposed three-dim space!) . Suppose you have an assemble of N identical particles, so the system admits as symmetry the symmetric group S_N. The Hilbert space of the system supports therefore a projective representation of this group; the two more important are ± 1, that is, the even permutations are represented identically.

I–3, Dimension 3

Now

- *Tensor objects are bosons*, that is, their wavefunction remains invariant under arbitrary permutation of identical (tensor) objects.
- *Spinor objects are fermions*, that is, their wavefunction changes sign under an odd permutation (= odd number of transpositions) of identical (spinor) objects.

Notice a sign change in the vector representative of a quantum state is not seen on the ray = state: that is why it is admitted in the first place. It is curious to realize that non-observability of the phase of the wavefunction leads to two (indeed, to more!) physically differentiable situations!
So fermions, so characteristic of the real world (ourselves!) are a double oddity: they compensate the -1 on the 2π rotation with another -1 in the interchange, whereas bosons are unsensitive to both -1!

I–4, Dimension 3

Sudarshan's old (1968) short proof of the spin-statistics connection in three dimensions stems from Schwinger's study of first order lagrangians (1951): the correctly antisymmetrized time-derivative part of the quantum lagrangian is, with $\chi = \{\chi_a\}$ the multicomponent field,

$$\mathcal{L} = \frac{1}{4}\left\{\chi^t K \frac{\partial}{\partial t}\chi - \left(\frac{\partial}{\partial t}\chi\right)^t K\chi\right\} - \dots$$

where K has to be an antihermitean numerical matrix, $K = -K^\dagger$ and $\dots$ means the rest of the lagrangian (space derivatives, Hamiltonian, etc; they do not play a role in what follows). But now there are clearly two possibilities:

- K is real and antisymmetric, $K = K^*$, $K = -K^t$,
- K is pure imaginary and symmetric, $K = -K^*$, $K = K^t$,

as already noticed by Schwinger.

I–5, Dimension 3

Now one imposes SO(3) invariance (it is enough!) on the K–term, and concludes the character of K from the Bose or Fermi type of the field χ: the symmetry type of the square of the χ field times the K–type has to produce the identity representation of SO(3), as this kinetic term has to be rotation invariant. Now, in our 3-space

- $\mathrm{D}_\ell \times \mathrm{D}_\ell$ contains the Id *irrep* in the symmetric part
- $\mathrm{D}_j \times \mathrm{D}_j$ contains the Id *irrep* in the antisymmetric part

Finally, the type of symmetry of K fixes the commutation / anticommutation relations through the equations

Bose: $2\imath\delta^3(\mathbf{x}-\mathbf{y}) = [\chi_a(\mathbf{x}), \chi_b(\mathbf{y})]\,\mathrm{K}_{ab}$, K real and antisymmetric

Fermi: $2\delta^3(\mathbf{x}-\mathbf{y}) = \{\chi_a(\mathbf{x}), \chi_b(\mathbf{y})\}\,\mathrm{K}_{ab}$, K real and symmetric

So you quantize conventionally bosons with commutators (antisymmetric) and fermions with anticommutators (symmetric).

I–6, Dimension 3

This is the essence of the Schwinger–Sudarshan simplified version of the original Pauli (1940) argument, much sophisticated by Burgoyne, Wightman, and others. It is easy to show the argument is consistent with Lorentz invariance, changing fixed times by a spacelike surface, etc.

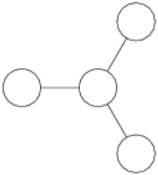
II–1, Dimension 8: triality

In the physics of present times extra dimensions are natural and frequent; most of people take it for granted the persistence of the Spin—Statistics connection regardless of the dimension, others just ignore the issue, arguing you would never find many particles in the extra dimension, it is difficult enough to find one!
But theorically one should face the issue: does the Spin–Statistics theorem holds in arbitrary dimension? Does the theorem depend also in the topology of the space? (e.g., whether the extra space is compact or not); here we address only the first question.

A simple argument would show at once that the usual proof would NOT follow in dimension eight: the relation of the covering group Spin(8) with the SO(8) group is still the same, namely 2:1, but the realization through the spin representations is not!

II–2. Dimension 8: triality

For space dimension 8 things are different: the Dynkin diagram for the SO(8) group enjoys triality:



so the three representations, the vector, $\Box$ and the two spinor, $\triangle$, are isomorphic: the Spin(8) group has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as center, and we have

$$\mathrm{Spin}(8) \longrightarrow \left\{ \begin{matrix} \triangle_L \\ \Box \\ \triangle_R \end{matrix} \right\} \longrightarrow \mathrm{SO}(8)/\mathbb{Z}_2$$

II–3. Dimension 8: triality

In fact, the triality group S_3 permutes the three real representaions of dimension 8. Therefore, the Id *irrep* is in the symmetric part of the square of either representation! :

- $\Box^2 = (1+35)_s + (28)_a$; 35: $\Box\Box \equiv [2]$
- $\triangle_L = (1+35)_s + (28)_a$; 35: selfdual 4–form $\equiv [1^4]$
- $\triangle_R = (1+35)_s + (28)_a$; 35: antiselfdual 4–form $\equiv [1^4]'$

Numerically $\Box\Box = 8 \cdot 9/2 - 1 =$ (anti-)selfdual,
$8 \cdot 7 \cdot 6 \cdot 5/2 \cdot 3 \cdot 4 \cdot 2$

Hence naive application of the argument leads to Bose statistics both for vector and for spinor representations! The oddity is that the Spin(8) group is always realized, in an irreducible representation, with $\mathbb{Z}_2$ kernel, hence the two spinor representations are isomorphic to the vector one!

II–4. Dimension 8: triality

Triality, that is, isomorphism between the three 8–dim *irreps*, happens only in dimension 8: the center of the spin group, which is **V** (Klein's Vierergruppe), admits S_3 as symmetry; this is always the case for dimension $4n$, but ONLY for $n = 2$ is the symmetry lifted to a symmetry of the full rotation group: the Dynkin diagrams for the $SO(4n)$ groups do not exhibit triality outside $n = 2$. So one would ask: *what about the connection in dimension outside 8?* This is addressed next.

III–1. Arbitrary dimension

The reason why dim $8n$ is different from 3 stems from Bott's periodicity for the orthogonal group, rather for the Spin groups: the spinor representation of SO(3) is symplectic, as Spin(3) $=$ SU(2) $=$ Sq(1), hence the scalar enters in the antisymmetric product. To see the behaviour in arbitrary dimension, it is enough to write the equivalences of the spin groups for low dimensions:

Spin(1)	Spin(2)	Spin(3)	Spin(4)	Spin(5)	Spin(6)	Spin(7,8)
$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	
O(1)	U(1)	Sq(1)	Sq$(1)^2$	Sq(2)	SU(4)	
$\mathbb{R}$	$\mathbb{C}$	$\longleftarrow$	$\mathbb{H}$	$\longrightarrow$	$\mathbb{C}$	$\mathbb{R}$

and the type repeats itself with (Bott) periodicity 8.

III–2. Arbitrary dimension

The final outcome is as follows: the type of the spin representation is

(A) For space dimensions $8k+4$, $8k+3$ and $8k+5$, **quaternionic type**

(B) For space dimensions $8k$, $8k-1$ and $8k+1$, **real type**.

(C) For space dimensions $4n+2$, **complex type**

As consequence:

- in cases (A) the usual connection obtains, spinors are fermions, and tensors are bosons.
- In case (B), spinors and tensors are bosons.
- In case (C), one can use the conventional statistics, but it is optional, not enforced (explained in the paper).

In all cases, tensor are always bosons (correspondence principle).

IV–1. Comments

The best lecture one makes of this argument is this: Supersymmetry imposes spinor–vector symmetry; this is best expressed precisely in 8 space dimensions!; e.g. in Superstring Theory. Even in M-Theory in $10 = (9, 1)$, the spinor type is real, and indeed the best SuSy is in this 11-dim Supergravity or membrane theory. As we descend, via compactification, to 4 spacetime dimensions, the supersymmetry is maintained, but quantization imposes anticommutation rules for fermions!

It would be nice to accept the following argument: supersymmetry is natural in 8 space dimensions, whatever the dimension of the actual, physical spacetime: 12, 11 or 10 in F–Theory, M–Theory or Superstrings; but this spinor/vector natural symmetry in 8-dim does not obtain in the compactifiaction process down to four-dimensional Minkowski space.

IV–2. Comments

Is it the Spin-Statistics connecion, that we undoubtedly see in our mundane 4-dim space, a reflection of the Susy symmetry upstairs? Perhaps is it too early to hinge too much in this possibility...

Statistics in two and one space dimensions is singular: the covering group of SO(2) is noncompact, which gives rise to anyons, and SO(1) = Id, there is no compelling reason to distinguish fermions from bosons: indeed, in some 1+1 quantum field theory, the fundamental field is bose, but the solitonic excitations behave as fermions.

Thanks for your attention.