The TT metric GW worksheet

Edoardo Milotti

November 21, 2022

To carry out the calculations we need to recall a few important expressions:

- Christoffel symbols
$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right) \tag{1}$$

- Riemann tensor
$$R^{\mu}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma} \Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} \Gamma^{\mu}_{\delta\gamma} \tag{2}$$

- Ricci tensor
$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} = \partial_\beta \Gamma^{\mu}_{\alpha\mu} - \partial_\mu \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\mu} \Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} \Gamma^{\mu}_{\delta\mu} \tag{3}$$

- metric tensor for the + polarization
$$[g_{\mu\nu}] = [\eta^{\mu\nu}] + [h^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 + h(t,z) & 0 & 0 \\ 0 & 0 & -1 - h(t,z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

- inverse metric tensor for the + polarization
$$[g^{\mu\nu}] = [\eta^{\mu\nu}] - [h^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 - h(t,z) & 0 & 0 \\ 0 & 0 & -1 + h(t,z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Next, using the *Diagonal metric worksheet* of T. A. Moore[1] we carry out the calculations that show that for the + polarization

$$R_{tt} = R_{zz} = \frac{1}{c^2} \left(h\ddot{h} + \frac{\dot{h}^2}{2} \right); \quad R_{xx} = R_{yy} = 0 \tag{6}$$

to second order in h.

Assuming a sinusoidal time dependence

$$h(t,z) = A\cos(\omega t - kz) = h^{TT}_{xx} = -h^{TT}_{yy} \tag{7}$$

[1] We must be careful that in Moore's text he uses the (-,+,+,+) signature while we use the (+,-,-,-) signature, and units such that $c = 1$.

and averaging, we find

$$\left\langle h\ddot{h} + \frac{\dot{h}^2}{2} \right\rangle = \omega^2 A^2 \left\langle -\cos^2(\omega t - kz) + \frac{1}{2}\sin^2(\omega t - kz) \right\rangle$$
$$= -\omega^2 A^2 \langle \cos(2\omega t - 2kz) \rangle - \frac{1}{2}\left\langle \dot{h}^2 \right\rangle = -\frac{1}{2}\left\langle \dot{h}^2 \right\rangle \quad (8)$$

It is also easy to see that the h-dependent part of the Ricci scalar vanishes.