Vectors and 1-forms

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Consider an orthonormal set of basis vectors $\{\mathbf{e}_i\}$, so that a generic vector can be expressed in the form

$$\mathbf{v} = v^i \mathbf{e}_i.$$

Next, consider the vector space of **linear functions** that act on these vectors, and let $\{\mathbf{e}^i\}$, be an orthonormal basis in this dual space such that a vector in the dual space can be written as a linear combination

$$\tilde{\mathbf{v}} = v_i \mathbf{e}^i,$$

and

$$\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j.$$

Now, thanks to linearity, the action of a generic element (function) of the dual space $\tilde{\mathbf{p}} = p_i \mathbf{e}^i$ on a vector $\mathbf{v} = v^j \mathbf{e}_j$ is easy to find:

$$\tilde{\mathbf{p}}(\mathbf{v}) = p_i \mathbf{e}^i \left(v^j \mathbf{e}_j \right) = p_i v^j \mathbf{e}^i(\mathbf{e}_j) = p_i v^j \delta^i_j = p_i v^i.$$

Note that the action of the basis vector of the dual space on a basis vector in the original space is also written as a dot product, and everything still works thanks to linearity:

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j,$$

and

$$\tilde{\mathbf{p}} \cdot \mathbf{v} = p_i v^j (\mathbf{e}^i \cdot \mathbf{e}_j) = p_i v^j \delta^i_j = p_i v^i,$$

where the dot product takes the priority over the sums.

Going back to the matrix definition of the metric tensor and dot product in SR, we find that

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}$$

and this extends to

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = g_{\mu\nu}$$

for Riemann manifolds.

Notice that when we take a linear orthogonal transformation A_{ij} of the set of basis vectors and its inverse A^{ij}, then we have

$$\mathbf{v} = v^i \mathbf{e}_i = v^i \delta^j_i \mathbf{e}_j = v^i A_{ik} A^{kj} \mathbf{e}_j = (v^i A_{ik})(A^{kj} \mathbf{e}_j)$$

so that coordinates transform in the opposite (i.e., according to the inverse transformation) with respect to basis vectors.

The vectors of coordinates in the original space are called *contravariant vectors*, while vectors of coordinates (which represent functions with vector arguments) of the dual space are called *1-forms* or *covariant vectors.*

When we consider a coordinate transformation $x'^{\mu} = x'^{\mu}(x)$, the linear approximant for infinitesimal changes is

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$

and $\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ is the transformation matrix for controvariant vectors. Its inverse $\frac{\partial x^{\mu}}{\partial x'^{\nu}}$ is the transformation matrix for covariant vectors.

Exercise: show that the line element in 3D flat space is given by the following expression, in spherical coordinates:

$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2.$

Exercise: use the previous result to find the metric tensor $g_{\mu\nu}$ for 3D flat space, in spherical coordinates.

Exercise: use the previous results to find the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ for the surface of a sphere of radius R.

Exercise: Show that the gradient of a scalar field is a covariant vector.