The transverse-traceless gauge

Edoardo Milotti

October 27, 2022

In empty space the wave equation with the Lorenz condition

$$\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{1}$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{2}$$

reduces to

$$\Box^2 \bar{h}_{\mu\nu} = 0 \tag{3}$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{4}$$

We have also found that a gauge transformation leads to a coordinate system that satisfies the Lorenz condition if

$$\Box^2 \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}, \tag{5}$$

and that there is an infinity of ξ_μ that satisfy this equation. This means that we assume the Lorenz gauge and still have residual freedom to constrain further the coordinate system.

We can spell out the wave equation

$$\frac{\partial^2 \bar{h}_{\mu\nu}}{\partial t^2} - c^2 \nabla^2 \bar{h}_{\mu\nu} = 0 \tag{6}$$

and look for solutions

$$\bar{h}^{\mu\nu} = \mathrm{Re}[A^{\mu\nu} \exp(i k_\alpha x^\alpha)] \tag{7}$$

These trial solutions must satisfy the following constraints:

- from the symmetry of the metric tensor: $A^{\mu\nu} = A^{\nu\mu}$
- from the wave equation: $k^\alpha k_\alpha = 0$, i.e., the wave 4-vector is null; this condition implies the usual dispersion relation, i.e., waves move with speed c
- from the Lorenz condition: $k_\nu A^{\mu\nu} = 0$

Here, we consider a wave propagating in the x^3 direction, so that the wave 4-vector is

$$k^\mu = (k, 0, 0, k); \quad k_\mu = (k, 0, 0, -k)$$

(the equality $k^0 = k^3$ follows from the fact that this is a null vector), with $k = \omega/c$. With this choice of k^μ, we find the following equality from the Lorenz condition

$$k A^{\mu 0} - k A^{\mu 3} = 0, \tag{8}$$

i.e.,

$$A^{\mu 0} = A^{\mu 3}, \tag{9}$$

and the A matrix writes

$$\begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix}$$

We see that the A matrix depends on just 6 quantities, $(A^{00}, A^{01}, A^{02}, A^{11}, A^{12}, A^{22})$, in line with what we expect from the application of the Lorenz condition (the symmetric tensor has 10 independent components, but the Lorenz condition implies 4 equalities, and hence the number of independent components is reduced to 6).

The Lorenz gauge determines a whole class of gauge transformations, so we need to further specify the coordinate transformation. We do this by choosing

$$\xi^\mu = -\mathrm{Re}[i\epsilon^\mu \exp(ik_\alpha x^\alpha)], \tag{10}$$

where k^μ is the same as in eq. (7), so that *this is a wave-dependent gauge transformation.* This trivially satisfies the Lorenz condition, and moreover from

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha \tag{11}$$

we find

$$A'^{\mu\nu} = A^{\mu\nu} - k^\nu \epsilon^\mu - \epsilon^\nu k^\mu + \eta^{\mu\nu} k_\alpha \epsilon^\alpha. \tag{12}$$

From eq. (12) and from the values of k^μ and $A^{\mu\nu}$ listed above, we find

$$A'^{00} = A^{00} - k^0\epsilon^0 - \epsilon^0 k^0 + \eta^{00} k_\alpha \epsilon^\alpha = A^{00} - k(\epsilon^0 + \epsilon^3) \tag{13}$$
$$A'^{01} = A^{01} - k^1\epsilon^0 - \epsilon^1 k^0 + \eta^{01} k_\alpha \epsilon^\alpha = A^{01} - k\epsilon^1 \tag{14}$$
$$A'^{02} = A^{02} - k^2\epsilon^0 - \epsilon^2 k^0 + \eta^{02} k_\alpha \epsilon^\alpha = A^{02} - k\epsilon^2 \tag{15}$$
$$A'^{11} = A^{11} - k^1\epsilon^1 - \epsilon^1 k^1 + \eta^{11} k_\alpha \epsilon^\alpha = A^{11} - k(\epsilon^0 - \epsilon^3) \tag{16}$$
$$A'^{12} = A^{12} - k^2\epsilon^1 - \epsilon^2 k^1 + \eta^{12} k_\alpha \epsilon^\alpha = A^{12} \tag{17}$$
$$A'^{22} = A^{22} - k^2\epsilon^2 - \epsilon^2 k^2 + \eta^{22} k_\alpha \epsilon^\alpha = A^{22} - k(\epsilon^0 - \epsilon^3) \tag{18}$$

In this way we have added four constraints (the values of ϵ^μ) and we can use eqs. (13)-(18) to reduce the independent components of A to just two. A^{12} remains unchanged by the gauge transformation, and $A^{21} = A^{12}$ by symmetry, and we can select just one more independent value. Setting

$$\epsilon^0 = (2A^{00} + A^{11} + A^{22})/4k \tag{19}$$
$$\epsilon^1 = A^{01}/k \tag{20}$$
$$\epsilon^2 = A^{02}/k \tag{21}$$
$$\epsilon^3 = (2A^{00} - A^{11} - A^{22})/4k \tag{22}$$

we find the following transformed A matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^{11} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Defining two *linear polarization matrices*

$$\epsilon_+^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \epsilon_\times^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we can write the generic polarization state

$$A'^{\mu\nu} = A_+\epsilon_+^{\mu\nu} + A_\times\epsilon_\times^{\mu\nu} \tag{23}$$

Both matrices are traceless and the 3 component vanishes (transverse propagation), so this choice of the gauge *the transverse-traceless gauge* (TT gauge). Since $\bar{h} = 0$, then $h = 0$ and therefore there is no difference between perturbation variables and trace-reversed perturbation variables.

In linearized gravity and in the TT gauge,

$$\Gamma^\alpha_{\mu\nu} \approx \frac{1}{2}\eta^{\alpha\gamma}\left(\partial_\mu h_{\gamma\nu} + \partial_\nu h_{\gamma\mu} - \partial_\gamma h_{\mu\nu}\right) \tag{24}$$

$$= \frac{1}{2}\eta^{\alpha\gamma}\left(k_\mu h_{\gamma\nu} + k_\nu h_{\gamma\mu} - k_\gamma h_{\mu\nu}\right); \tag{25}$$

considering this expression, it is easy to see that

$$\Gamma^\mu_{00} = 0 \quad \Gamma^\mu_{0\nu} = \frac{1}{2}\partial_0 h^\mu_\nu.$$

Using these results, and considering a particle initially at rest, so that its initial 4-velocity is $\dot{x}^\mu = (c, 0, 0, 0)$, the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{26}$$

becomes

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\Gamma^\alpha_{00}c^2 = 0. \tag{27}$$

This means that the velocity remains constant and equal to its initial value (particle at rest). In other words, in the TT gauge, a cloud of particles at rest has geodesics with constant spatial coordinate: therefore, the small spacelike vectors $\xi^\mu = (0, \xi^1, \xi^2, \xi^3)$ that mark the separations between nearby particles in the cloud remain constant.

However, the spatial separation d is *not* constant:

$$d^2 = -g_{\mu\nu}\xi^\mu\xi^\nu = -(\eta_{ij} + h_{ij})\xi^i\xi^j = (\delta_{ij} - h_{ij})\xi^i\xi^j = \xi_i\xi^i - h_{ij}\xi^i\xi^j$$

$$\approx \left(\xi_i - \frac{1}{2}h_{ik}\xi^k\right)\left(\xi^i - \frac{1}{2}h^i_k\xi^k\right) \tag{28}$$

The new variables $\xi^i - \frac{1}{2}h^i_k\xi^k$ mark the correct spatial separation. Note that in the TT gauge, there is no shift in the 3 direction (the propagation direction), again showing that the wave is transverse.

When we take one of these particles, originally in $(\xi^1, \xi^2, 0)$ as position marker, we see that with a passing wave with amplitude $A_+\epsilon_+^{\mu\nu}$ its position is

$$x^1 = \xi^1 - \frac{A_+}{2}\cos\omega t\;\xi^1 \tag{29}$$

$$x^2 = \xi^2 + \frac{A_+}{2}\cos\omega t\;\xi^2 \tag{30}$$

$$x^3 = 0 \tag{31}$$

This means that in the $\xi^3 = 0$ plane, a ring of N equally spaced reference masses at radius R and angular position $\theta_n = 2\pi n/N$ has separation

$$r_n^2 = R^2\left(1 - \frac{A_+}{2}\cos\omega t\right)^2 \cos^2\theta_n + R^2\left(1 + \frac{A_+}{2}\cos\omega t\right)^2 \sin^2\theta_n \tag{32}$$

$$\approx R^2\left[(1 - A_+\cos\omega t)\cos^2\theta_n + (1 + A_+\cos\omega t)^2\sin^2\theta_n\right] \tag{33}$$

$$= R^2\left[1 - A_+(\cos^2\theta_n - \sin^2\theta_n)\cos\omega t\right] \tag{34}$$

$$= R^2\left(1 - A_+\cos 2\theta_n\;\cos\omega t\right), \tag{35}$$

and finally

$$r_n \approx R\left(1 - \frac{A_+}{2}\cos 2\theta_n \cos\omega t\right). \tag{36}$$

We see that the perturbation variable represents a relative deformation of the distance between test masses; in the theory of elasticity this is called a *strain*.