Special Relativity and notation

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Special Relativity (SR) is a necessary prerequisite of this course and is taken for granted. In this initial chapter I set out the conventions that are followed in the lecture notes.

- For clarity, I always include all the physical constant, therefore c, G, etc., are always spelled out explicitly.
- I assume the $(+,-,-,-)$ signature, i.e., the space-time interval is defined by $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.
- Latin letters indicate space variables in 3D space or in a generic n-dimensional space, while greek letters denote space-time variables in 4D space.
- The previous items imply that the Minkowski metric is specified by the following metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

  i.e., $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

- Indexes follow the Einstein convention: repeated indexes imply a summation. For example, when we denote the inverse metric tensor with $\eta_{\mu\nu}$, and notice that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

  then we conclude that the matrix representation of $\eta_{\mu\nu}$ is the same as that of its inverse $\eta^{\mu\nu}$, and clearly $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu$.

- Lorentz transformations along a specific axis (x, in this case) are defined by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

  with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, as usual.

- Proper time τ is time in the rest frame of the observer. Therefore, if something is at rest in the observer's frame (i.e., $dx = dy = dz = 0$), $ds^2 = c^2 d\tau^2$.

Problem: express the space-time interval ds^2 of SR space-time in spherical coordinates. (Solution: $ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2$).

To introduce dynamics, we must define first a 4-vector velocity. This is done as follows:

$$U^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} cdt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix}.$$

Since

$$\frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\gamma; \quad \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma v^i$$

then

$$U^\mu = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}.$$

Here we note in passing that

$$\eta_{\mu\nu}U^\mu U^\nu = \gamma^2c^2 - \gamma^2v^2 = c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = c^2.$$

4-momentum is defined as follows

$$P^\mu = mU^\mu = \begin{pmatrix} m\gamma c \\ \gamma m\mathbf{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix},$$

therefore

$$\eta_{\mu\nu}P^\mu P^\nu = \frac{E^2}{c^2} - p^2 = m^2c^2,$$

or also

$$E^2 = m^2c^4 + c^2p^2.$$

Finally, the special relativistic extension of Newton's second law is

$$F^\mu = \frac{dP^\mu}{d\tau}.$$