Symmetries of the Riemann and Ricci tensors

Edoardo Milotti

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Recall the expression of the Riemann tensor as a sum of Christoffel symbols and their derivatives

$$R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec}, \tag{1}$$

so that, lowering the d index, and evaluating the expression at the origin of a LIF in spacetime, we find

$$R_{\mu\alpha\beta\gamma} = g_{\mu\delta}\left(\partial_\beta \Gamma^\delta_{\alpha\gamma} - \partial_\gamma \Gamma^\delta_{\alpha\beta}\right) \tag{2}$$

(the last two terms in (1) vanish at the origin of the LIF, because they are made up of derivatives of the metric tensor). Next, notice that at the origin of the LIF the Christoffel symbols vanish and

$$\partial_\beta \Gamma^\delta_{\alpha\gamma} = \frac{1}{2} g^{\delta\sigma}\left(\partial_\beta\partial_\alpha g_{\sigma\gamma} + \partial_\beta\partial_\gamma g_{\sigma\alpha} - \partial_\beta\partial_\sigma g_{\alpha\gamma}\right)$$

and

$$\partial_\gamma \Gamma^\delta_{\alpha\beta} = \frac{1}{2} g^{\delta\sigma}\left(\partial_\gamma\partial_\alpha g_{\sigma\beta} + \partial_\gamma\partial_\beta g_{\sigma\alpha} - \partial_\gamma\partial_\sigma g_{\alpha\beta}\right)$$

so that

$$\begin{aligned}
R_{\mu\alpha\beta\gamma} &= \frac{1}{2} g^{\delta\sigma} g_{\mu\delta}\left(\partial_\beta\partial_\alpha g_{\sigma\gamma} - \partial_\beta\partial_\sigma g_{\alpha\gamma} - \partial_\gamma\partial_\alpha g_{\sigma\beta} + \partial_\gamma\partial_\sigma g_{\alpha\beta}\right) && (3)\\
&= \frac{1}{2} \delta^\sigma_\mu\left(\partial_\beta\partial_\alpha g_{\sigma\gamma} - \partial_\beta\partial_\sigma g_{\alpha\gamma} - \partial_\gamma\partial_\alpha g_{\sigma\beta} + \partial_\gamma\partial_\sigma g_{\alpha\beta}\right) && (4)\\
&= \frac{1}{2}\left(\partial_\beta\partial_\alpha g_{\mu\gamma} - \partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\gamma\partial_\alpha g_{\mu\beta} + \partial_\gamma\partial_\mu g_{\alpha\beta}\right) && (5)\\
&= \frac{1}{2}\left(\partial_\alpha\partial_\beta g_{\gamma\mu} - \partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\mu} + \partial_\gamma\partial_\mu g_{\alpha\beta}\right) && (6)
\end{aligned}$$

(the last step is just an alphabetical reordering of symmetrical indexes).

Now we use expression (6) to prove a handful of useful symmetries of the Riemann tensor

- exchanging the first two indexes, μ and α, and using the symmetry of the metric tensor, we find

$$R_{\alpha\mu\beta\gamma} = \frac{1}{2}\left(\partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\alpha\partial_\beta g_{\gamma\mu} - \partial_\gamma\partial_\mu g_{\alpha\beta} + \partial_\alpha\partial_\gamma g_{\beta\mu}\right) = -R_{\mu\alpha\beta\gamma}; \tag{7}$$

- exchanging the third and fourth indexes, β and γ, we find

$$R_{\mu\alpha\gamma\beta} = \frac{1}{2}\left(\partial_\alpha\partial_\gamma g_{\beta\mu} - \partial_\gamma\partial_\mu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\gamma\mu} + \partial_\beta\partial_\mu g_{\alpha\gamma}\right) = -R_{\mu\alpha\beta\gamma}; \tag{8}$$

- exchanging the first and third indexes (μ and β) AND the second and fourth indexes (α and γ), we find

$$R_{\beta\gamma\mu\alpha} = \frac{1}{2}\left(\partial_\gamma\partial_\mu g_{\alpha\beta} - \partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\mu} + \partial_\alpha\partial_\beta g_{\gamma\mu}\right) = R_{\mu\alpha\beta\gamma}; \tag{9}$$

- considering the cyclic permutations of the last three indexes, $\beta\gamma\alpha$ and $\gamma\alpha\beta$ in addition to the original $\alpha\beta\gamma$, we find

$$R_{\mu\beta\gamma\alpha} = \frac{1}{2}\left(\partial_\beta\partial_\gamma g_{\alpha\mu} - \partial_\gamma\partial_\mu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\gamma\mu} + \partial_\alpha\partial_\mu g_{\beta\gamma}\right), \tag{10}$$

  and

$$R_{\mu\gamma\alpha\beta} = \frac{1}{2}\left(\partial_\alpha\partial_\gamma g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\mu} + \partial_\beta\partial_\mu g_{\alpha\gamma}\right), \tag{11}$$

  therefore

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0. \tag{12}$$

- now recall that at the origin of the LIF the Christoffel symbols vanish, so that the covariant derivative equals the common derivative, and we find (also reordering symbols in alphabetical order, using symmetry)

$$\nabla_\sigma R_{\mu\alpha\beta\gamma} = \frac{1}{2}\left(\partial_\alpha\partial_\beta\partial_\sigma g_{\gamma\mu} - \partial_\beta\partial_\mu\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma\partial_\sigma g_{\beta\mu} + \partial_\gamma\partial_\mu\partial_\sigma g_{\alpha\beta}\right); \tag{13}$$

  similarly, when we cyclically permute σ, β and γ, we find

$$\nabla_\beta R_{\mu\alpha\gamma\sigma} = \frac{1}{2}\left(\partial_\alpha\partial_\gamma\partial_\beta g_{\sigma\mu} - \partial_\beta\partial_\gamma\partial_\mu g_{\alpha\sigma} - \partial_\alpha\partial_\beta\partial_\sigma g_{\gamma\mu} + \partial_\beta\partial_\mu\partial_\sigma g_{\alpha\gamma}\right); \tag{14}$$

  and

$$\nabla_\gamma R_{\mu\alpha\sigma\beta} = \frac{1}{2}\left(\partial_\alpha\partial_\gamma\partial_\sigma g_{\beta\mu} - \partial_\gamma\partial_\mu\partial_\sigma g_{\alpha\beta} - \partial_\alpha\partial_\beta\partial_\gamma g_{\mu\sigma} + \partial_\beta\partial_\gamma\partial_\mu g_{\alpha\sigma}\right); \tag{15}$$

  Finally,

$$\nabla_\sigma R_{\mu\alpha\beta\gamma} + \nabla_\beta R_{\mu\alpha\gamma\sigma} + \nabla_\gamma R_{\mu\alpha\sigma\beta} = 0 \tag{16}$$

  The last equation is called *Bianchi identity.*

While these equations have been proved true at the origin the LIF, their tensor form is independent of the particular choice of frame of reference, and therefore they are always true.