# Overview of Riemann manifolds

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October 22, 2022

The equivalence principle (in essence, the indistinguishability between gravitational pull and dynamical acceleration) implies that the effects of gravity must be associated with *a curvature of space-time*. Mathematically, this means that we must turn to the manifolds studied by Riemann.

- A Riemann manifold can be loosely described as a smoothly curved space that is locally flat.
- A manifold is $n$-dimensional when the position of a point is specified by $n$ coordinates.
- Not all continuous spaces are manifolds. E.g., a one-dimensional line emerging from a plane is not a manifold; two cones joined at the apex are not a manifold (parts of these objects are not locally Euclidean).
- A manifold can be embedded in a larger space and display **extrinsic curvature**.
- A manifold can exist without any embedding at all and display **intrinsic curvature**.
- A manifold can display both extrinsic and intrinsic curvature.
- Manifolds exist that have extrinsic curvature and no intrinsic curvature.
- Riemann discovered that the *metric tensor* $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$ contains all the information necessary to describe a manifold.
- The metric tensor is symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$.
- Riemann manifolds have positive-definite metrics; the metric of space-time is not positive-definite, and it is described by a pseudo-Riemann manifold.
- The line element of a Riemann manifold is given by $d\ell^2 = g_{ij} dx^i dx^j$.
- The description of Riemann manifolds requires tensor calculus.
- The manifolds of General Relativity are pseudo-Riemann because the metric is not positive-definite.