Noise sources-2

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In this second handout on noise sources, I explore quantum noise, i.e. photon shot noise and quantum radiation pressure noise, I introduce the Standard Quantum Limit (SQL), and briefly describe the methods to mitigate it.

Shot noise

Let N_γ be the average number of photons with energy $\hbar\omega_\gamma$ that reach the detector from the laser source during time T. Then, the average power measured during this observation time is

$$P = \frac{1}{T} N_\gamma \hbar\omega_\gamma \tag{1}$$

It is fair to assume that the actual number of photons is a Poisson variate with variance equal to the average N_γ, so that the standard deviation is $\Delta N_\gamma = \sqrt{N_\gamma}$. The corresponding fluctuation of the measured power is

$$\Delta P_{\text{shot}} = \frac{1}{T}\sqrt{N_\gamma}\hbar\omega_\gamma = \left(\frac{\hbar\omega_\gamma}{T}P\right)^{1/2} \tag{2}$$

Now, we want to assess the effect of this noise on GW measurements by comparing it with the power received by the photodiode when a GW signal is present, and we use an oversimplified model of GW interferometer, which consists in a simple Michelson interferometer. Then, in the absence of signal, one finds (see eq. (37) in the Appendix) that the output power is

$$P = \frac{P_0}{2}[1 - \cos(2k_\gamma \Delta L)] = P_0 \sin^2(k_\gamma \Delta L) \tag{3}$$

so that

$$\Delta P_{\text{shot}} = \left(\frac{\hbar\omega}{T}P_0\right)^{1/2} |\sin(k_\gamma \Delta L)| \tag{4}$$

When a GW signal is present, we find that the power change is (see eq. (38) in the Appendix)

$$\Delta P_{\text{GW}} = P_0 \sin(2k_\gamma \Delta L)\frac{4\pi L}{\lambda_\gamma} h_0 \tag{5}$$

and therefore the power signal-to-noise ratio (SNR) is

$$\frac{S}{N} = \frac{\Delta P_{\text{GW}}}{\Delta P_{\text{shot}}} = \left(\frac{TP_0}{\hbar\omega_\gamma}\right)^{1/2} |\cos(k_\gamma \Delta L)| \frac{4\pi L}{\lambda_\gamma} h_0 \tag{6}$$

When we take as a reference the (unrealistic) value $|\cos(k_\gamma \Delta L)| = 1/\sqrt{2}$ the last formula reduces to

$$\frac{S}{N} = \left(\frac{TP_0}{2\hbar\omega_\gamma}\right)^{1/2} \frac{4\pi L}{\lambda_\gamma} h_0 = \left(\frac{T}{S_{\rm shot}(\omega)}\right)^{1/2} h_0 \tag{7}$$

In general, we notice that the higher the power, the larger the SNR: powerful lasers reduce the impact of shot noise! We also see that in this case the amplitude spectral density of shot noise is[1]

$$S_{\rm shot}^{1/2}(\omega) = \left(\frac{2\hbar\omega_\gamma}{P_0}\right)^{1/2} \frac{\lambda_\gamma}{4\pi L} \tag{8}$$

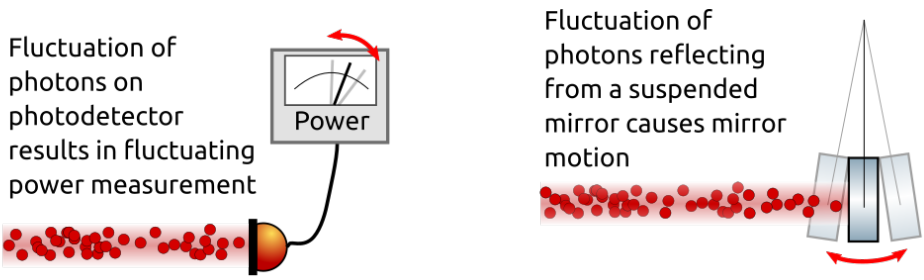

Figure 1: Pictorial representation of shot noise and radiation pressure noise.

Radiation pressure noise

Using the same formalism we can compute the radiation pressure noise. This is the noise due to the fluctuations of radiation pressure on the mirrors of the interferometer, due to the uneven flow of photons, and it is clearly related to the shot noise.

When a laser beam impinges on a mirror, there is a flow of photons that transfer momentum to the mirror itself. Since the momentum transferred to the mirror by each reflected photon

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[1] We understand the role of T a follows. In a stationary zero-mean process $x(t)$, the spectral density is related to the variance of the process by the Wiener-Kintchine theorem

$$S(f) = \int_{-\infty}^{+\infty} \langle x(0)x(t)\rangle e^{-2\pi i f t} dt$$

In the case of a white noise $R(t) = \sigma^2\delta(t)$, therefore – approximating the delta function with a rectangular pulse of width T, which is the sampling time of the order of 10^{-3} s – we find

$$S(f) \approx \frac{\sigma^2}{T}$$

in the case of a two-sided power spectral density, or also

$$S(f) \approx \frac{\sigma^2}{2T}$$

for a one sided power spectral density.

is $2\hbar\omega_\gamma/c$ the total momentum transferred by N_γ photons in a time T is $2N_\gamma\hbar\omega_\gamma/c$ and the corresponding force exerted on the mirror is

$$F = N_\gamma \frac{2\hbar\omega_\gamma}{cT} = \frac{2P_0}{c}. \tag{9}$$

This means that the fluctuation of the force due to the uneven flow of photons is

$$\Delta F = \frac{2}{c}\Delta P_{\text{shot}} = 2\left(\frac{\hbar\omega_\gamma}{c^2 T}P_0\right)^{1/2} \tag{10}$$

which does not depend on the GW frequency (ω) and therefore is a white noise with spectral density

$$S_F(\omega) = (\Delta F)^2 \times 2T \tag{11}$$

where the T factor accounts for the finite observation time (replacement for the delta function) and the factor 2 means that we are using a one-sided spectral density.

Now we can use a formula that we derived in the previous handout on Noise Sources

$$S_x(\omega) = \frac{S_n(\omega)}{(k - m\omega^2)^2 + \gamma^2\omega^2} \tag{12}$$

which we apply to the fluctuations of the mirror position and to the force noise. We also assume that for our mirrors the elastic constant and the friction constant are negligible, so that

$$S_x(\omega) \approx \frac{S_F(\omega)}{m^2\omega^4} \tag{13}$$

or also, when we take the amplitude spectral density

$$S_x^{1/2}(\omega) \approx \frac{S_F^{1/2}(\omega)}{m\omega^2} = \frac{\Delta F\sqrt{2T}}{m\omega^2} = 2\left(2\frac{\hbar\omega_\gamma}{c^2}P_0\right)^{1/2}\frac{1}{m\omega^2} \tag{14}$$

The Standard Quantum Limit (SQL)

In the previous sections we found that the amplitude disturbance due to shot noise scales as $1/\sqrt{P_0}$ and has a white spectrum, while the disturbance due to radiation pressure scales as $\sqrt{P_0}/m\omega^2$ (therefore it is not a white noise). The actual sensitivity of the apparatus must take into account the response to a GW, and this means that we must multiply times the transfer function of the interferometer to obtain the sensitivity curve; this operation transforms the high-frequency part of the spectrum which is mostly contributed by shot noise into a non-flat, degrading sensitivity (the S/N decreases). Taking both terms into account shows that there is a minimum overall noise that corresponds to the so-called Standard Quantum Limit. The SQL depends on laser power, as indicated by the formulas and pictorially shown in figure 2.

Squeezing

The SQL can be beaten by squeezing the laser light. The basics of squeezing of light are covered in the tutorial by Saleh and Teich, *Squeezed states of light*, **1** Quantum Opt. (1989) 153. The use of light squeezing to beat the SQL is explained by S. Hild, *A Basic Introduction to Quantum Noise and Quantum-Non-Demolition Techniques*, in M. Bassan (ed.) *Advanced Interferometers and the Search for Gravitational Waves, Lectures from the First VESF School on Advanced Detectors for Gravitational Waves* Springer (2014).

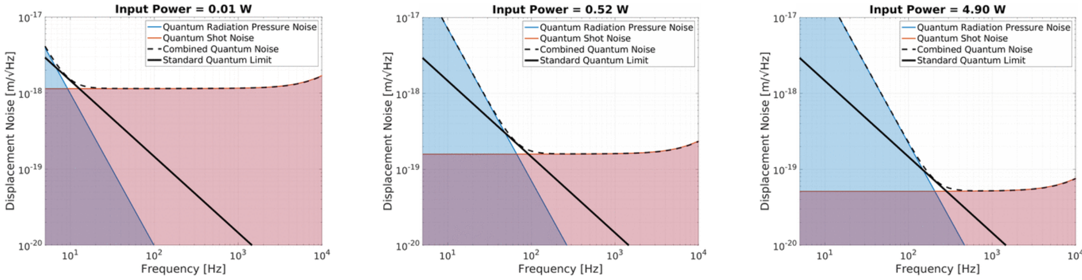

Figure 2: Pictorial representation of the interplay of shot noise and radiation pressure noise that yields the SQL.

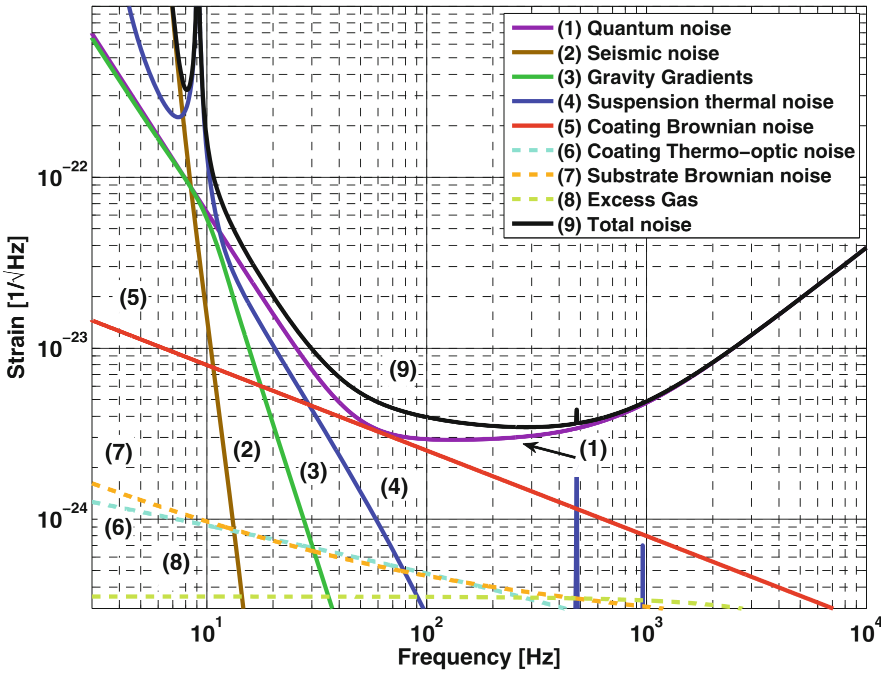

Figure 3: Noise budget of the advanced LIGO broadband configuration as described in R.Abbott et al., AdvLIGO Interferometer Sensing and Control Conceptual Design, Technical note LIGO-T070247-01-I (2008)

Appendix 1: time domain analysis of the effect of a GW on a Michelson interferometer

To make things simple, we consider a basic Michelson interferometer without FP cavities in the arms. A photon traveling in the x direction, along one of the axes of a + polarized GW, follows a null line element so that

$$cdt = \left(1 + \frac{1}{2}h(t)\right) dx. \tag{15}$$

where

$$h(t) = h_0 \cos \omega t \tag{16}$$

We can separate variables as follows

$$dx = \frac{cdt}{1 + \frac{1}{2}h(t)} \approx cdt \left(1 - \frac{1}{2}h(t)\right) \tag{17}$$

and integrate over the first pass, from the beamsplitter to the far mirror in the x direction

$$L_x = c \int_{t_0}^{t_1} \left(1 - \frac{1}{2}h(t)\right) dt = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} h(t)dt. \tag{18}$$

The integration along the second pass, from the far mirror back to the beamsplitter is similar, with a sign reversal for the velocity and with the reversal of the integration limits for the x integration,

$$-L_x = -c \int_{t_1}^{t_2} \left(1 - \frac{1}{2}h(t)\right) dt = -c(t_2 - t_1) + \frac{c}{2} \int_{t_0}^{t_1} h(t)dt. \tag{19}$$

Therefore, subtracting the second equation from the first we obtain

$$2L_x = c(t_2 - t_0) - \frac{c}{2} \int_{t_0}^{t_2} h(t)dt, \tag{20}$$

i.e.,

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2} \int_{t_0}^{t_2} h(t)dt, \tag{21}$$

The integral is already order h, therefore we can integrate with $t_2 \approx t_0 + 2L_x/c$, i.e.,

$$t_2 - t_0 \approx \frac{2L_x}{c} + \frac{h_0}{2} \int_{t_0}^{t_0+2L_x/c} \cos \omega t \, dt = \frac{2L_x}{c} + \frac{h_0}{2\omega} \left\{ \sin \left[\omega \left(t_0 + \frac{2L_x}{c} \right) \right] - \sin \omega t_0 \right\}. \tag{22}$$

The expression in curly brackets can be simplified to

$$2 \sin \frac{L_x \omega}{c} \cos \left[\omega \left(t_0 + \frac{L_x}{c} \right) \right],$$

and therefore we find

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{h_0}{\omega} \sin \frac{L_x \omega}{c} \cos \left[\omega \left(t_0 + \frac{L_x}{c} \right) \right] \tag{23}$$

$$= \frac{2L_x}{c} + \frac{L_x}{c} \frac{\sin(L_x \omega / c)}{L_x \omega / c} \, h\left(t_0 + \frac{L_x}{c} \right). \tag{24}$$

Along the y direction the sign of h is reversed so that

$$t_2 - t_0 = \frac{2L_y}{c} - \frac{L_y}{c}\frac{\sin(L_y\omega/c)}{L_y\omega/c}\,h\left(t_0 + \frac{L_y}{c}\right). \tag{25}$$

We combine the x and y information as follows: we require that the time of arrival of the two beams on the beamsplitter be the same $t = t_2$, and using the results above, we find the departure time t_0. First, we note that the argument of the strain

$$t_0 + \frac{L_x}{c} \approx t - \frac{L_x}{c}, \tag{26}$$

then,

$$t_0^{(x)} = t - \frac{2L_x}{c} - \frac{L_x}{c}\mathrm{sinc}(L_x\omega/c)\,h\left(t - \frac{L_x}{c}\right) \tag{27}$$

$$t_0^{(y)} = t - \frac{2L_y}{c} + \frac{L_y}{c}\mathrm{sinc}(L_y\omega/c)\,h\left(t - \frac{L_y}{c}\right). \tag{28}$$

The spatial position is the same, so the relative phase between the x and y electric fields of the laser beam depends on time only. In particular, we find

$$E^{(x)}(t) = -\frac{E_0}{2}\exp\left(-i\omega_\gamma t_0^{(x)}\right) = -\exp\left[-i\omega_\gamma\left(t - \frac{2L_x}{c} - \frac{L_x}{c}\mathrm{sinc}(L_x\omega/c)\,h\left(t - \frac{L_x}{c}\right)\right)\right] \tag{29}$$

$$E^{(y)}(t) = +\frac{E_0}{2}\exp\left(-i\omega_\gamma t_0^{(y)}\right) = +\exp\left[-i\omega_\gamma\left(t - \frac{2L_y}{c} + \frac{L_y}{c}\mathrm{sinc}(L_y\omega/c)\,h\left(t - \frac{L_y}{c}\right)\right)\right]. \tag{30}$$

where ω_γ is the (angular) frequency of the laser light.

We introduce the mean length $L = (L_x + L_y)/2$ and the difference $\Delta L = L_x - L_y$, so that $2L_x = 2L + \Delta L$ and $2L_y = 2L - \Delta L$. Then the phases become

$$\phi_x = \omega_\gamma\left[t - \frac{2L + \Delta L}{c}\right] - \omega_\gamma\frac{L + \Delta L/2}{c}\mathrm{sinc}[(L + \Delta L/2)\omega/c]\,h\left(t - \frac{L + \Delta L/2}{c}\right) \tag{31}$$

$$\approx \omega_\gamma\left[t - \frac{2L}{c}\right] - k_\gamma\Delta L - \left[h_0 k_\gamma L\,\mathrm{sinc}\left(\omega\frac{L}{c}\right)\right]\cos[\omega(t - L/c)] \tag{32}$$

and

$$\phi_y = \omega_\gamma\left[t - \frac{2L - \Delta L}{c}\right] + \omega_\gamma\frac{L - \Delta L/2}{c}\mathrm{sinc}[(L - \Delta L/2)\omega/c]\,h\left(t - \frac{L - \Delta L/2}{c}\right) \tag{33}$$

$$\approx \omega_\gamma\left[t - \frac{2L}{c}\right] + k_\gamma\Delta L + \left[h_0 k_\gamma L\,\mathrm{sinc}\left(\omega\frac{L}{c}\right)\right]\cos[\omega(t - L/c)] \tag{34}$$

Thus, the total electric field is

$$E_{\mathrm{tot}}(t) = E^{(x)}(t) + E^{(y)}(t) = \frac{E_0}{2}\left[-e^{-i\phi_x} + e^{-i\phi_y}\right] \tag{35}$$

$$= iE_0\exp\omega_\gamma\left[-i\omega_\gamma\left(t - \frac{2L}{c}\right)\right]\sin\left[k_\gamma\Delta L + \Delta\phi(t)\right] \tag{36}$$

where E_0 is the amplitude of the field as at the beamsplitter input, and

$$\Delta\phi(t) = \left[h_0 k_\gamma L \,\mathrm{sinc}\left(\omega \frac{L}{c}\right)\right] \cos[\omega(t - L/c)]$$

is the GW-dependent phase. Notice also that the $k_\gamma \Delta L$ term is associated with the Schnupp asymmetry.

From eq. (36) we find that the output power is

$$P = P_0 \sin^2 \left[k_\gamma \Delta L + \Delta\phi(t)\right] = \frac{P_0}{2} \left\{1 - \cos\left[2k_\gamma \Delta L + 2\Delta\phi(t)\right]\right\} \tag{37}$$

and that the change in output power due to a GW signal is

$$\Delta P = P_0 \sin(2k_\gamma \Delta L)\Delta\phi(t) \leq 2P_0 \sin(2k_\gamma \Delta L) h_0 k_\gamma L = P_0 \sin(2k_\gamma \Delta L) \frac{4\pi L}{\lambda_\gamma} h_0 \tag{38}$$

Appendix 2: Stokes treatment of reflection and refraction

Note: this text is adapted from E. Hecht, Optics, 4th ed., Addison-Wesley (2002).

Figure 4 shows rays reflected and refracted, and the way the refraction and transmission coefficients modify amplitudes. Primed coefficients corresponds to incidence from the higher density medium. The middle panel shows the time-reversed ray configuration. The right panel shows the superposition of left and middle panel.

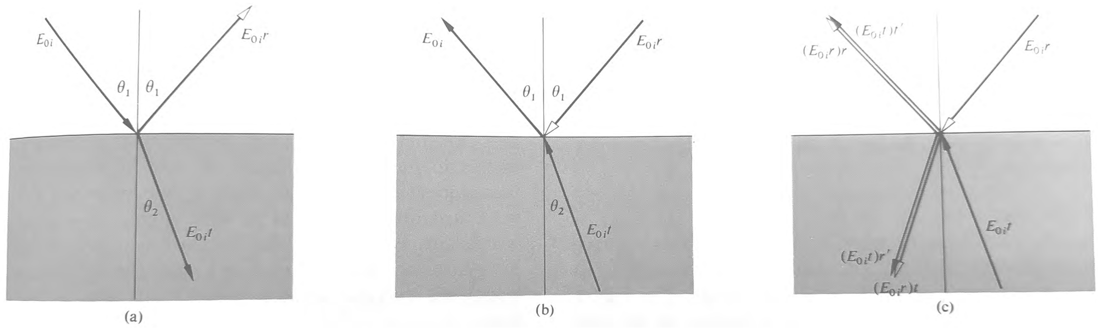

Figure 4: Figure illustrating the Stokes treatment of reflection and transmission coefficients.

From the superposition diagram we find

$$E_{0i} t t' + E_{0i} r r = E_{0i} \tag{39}$$

$$E_{0i} r t + E_{0i} t r' = 0 \tag{40}$$

And therefore

$$r' = -r \tag{41}$$

$$t t' = 1 - r^2 \tag{42}$$

This treatment holds for near normal refraction, and it can be modified accordingly for non-normal incidence.