Example of Newtonian limit

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October 22, 2022

The Newtonian limit of General Relativity is defined by the conditions

- all motions are slow ($v \ll c$)
- gravitational fields are weak
- gravitational fields are static

From these conditions we can retrieve the usual equation of motion in a gravitational field

$$\frac{d^2 x^i}{dt^2} = -\partial_i \Phi,$$

where Φ is the nonrelativistic gravitational potential. To this end, we note first that by the assumption that all motions are slow

$$\frac{dx^i}{d\tau} \ll c \quad \Rightarrow \quad \frac{dx^i}{d(c\tau)} \ll 1 \quad \text{and} \quad \frac{dt}{d\tau} \approx 1$$

then the only non-negligible terms in the geodesic equations are those with the Γ^α_{00} Christoffel symbols

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \quad \Rightarrow \quad \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{00} \left(\frac{cdt}{d\tau} \right)^2 \approx 0, \tag{1}$$

Using the expression

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left(\frac{\partial g_{\nu\gamma}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\nu} \right), \tag{2}$$

and the assumption of static field, so that the time derivatives vanish, we find

$$\Gamma^\alpha_{00} = \frac{1}{2} g^{\alpha\nu} \left(-\frac{\partial g_{00}}{\partial x^\nu} \right) = -\frac{1}{2} g^{\alpha\nu} \frac{\partial g_{00}}{\partial x^\nu}, \tag{3}$$

Finally, from the assumption that fields are weak, we consider the metric tensor to be approximately equal to $\eta_{\mu\nu}$ but for a small perturbation $h_{\mu\nu}$ (such that $|h_{\mu\nu}| \ll 1$):

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

The inverse metric tensor has a similar expression

$$g^{\mu\nu} \approx \eta^{\mu\nu} + h'^{\mu\nu} \tag{5}$$

and by contracting it with the metric tensor we obtain

$$\delta^\mu_\nu = g_{\mu\alpha} g^{\alpha\nu} \approx (\eta_{\mu\alpha} + h_{\mu\alpha})(\eta^{\alpha\nu} + h'^{\alpha\nu}) \approx \delta^\mu_\nu + \eta_{\mu\alpha} h'^{\alpha\nu} + h_{\mu\alpha} \eta^{\alpha\nu}, \tag{6}$$

therefore

$$\eta_{\mu\alpha} h'^{\alpha\nu} = -h_{\mu\alpha} \eta^{\alpha\nu}, \tag{7}$$

i.e.,

$$h'^{\mu\nu} = -\eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}. \tag{8}$$

Accordingly, from

$$h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta} = -h'^{\mu\nu} \tag{9}$$

we obtain the expression for the inverse metric tensor

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \tag{10}$$

Going back to eq. (3), we find that these definitions imply

$$\Gamma^{\alpha}_{00} \approx -\frac{1}{2} \eta^{\alpha\nu} \frac{\partial h_{00}}{\partial x^{\nu}}, \tag{11}$$

and the geodesic equation

$$\frac{d^2 x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{00} = \frac{d^2 x^{\alpha}}{d\tau^2} - \frac{c^2}{2} \eta^{\alpha\nu} \frac{\partial h_{00}}{\partial x^{\nu}} \approx 0, \tag{12}$$

The field is static, therefore

$$\frac{d^2 x^0}{d\tau^2} - \frac{c^2}{2} \eta^{0\nu} \frac{\partial h_{00}}{\partial x^{\nu}} = \frac{d^2 x^0}{d\tau^2} - \frac{c^2}{2} \frac{\partial h_{00}}{\partial x^0} \approx 0 \tag{13}$$

i.e.,

$$\frac{d^2 t}{d\tau^2} \approx \frac{c}{2} \frac{\partial h_{00}}{\partial t} = 0 \tag{14}$$

so that $\frac{dt}{d\tau}$ is constant.

Moving now to the space coordinates, we find

$$\frac{d^2 x^i}{d\tau^2} - \frac{c^2}{2} \eta^{i\nu} \frac{\partial h_{00}}{\partial x^{\nu}} = \frac{d^2 x^i}{d\tau^2} + \frac{c^2}{2} \frac{\partial h_{00}}{\partial x^i} \approx 0, \tag{15}$$

Finally, setting $h_{00} = 2\Phi/c^2$ we obtain

$$\frac{d^2 x^i}{d\tau^2} \approx -\frac{\partial \Phi}{\partial x^i}, \tag{16}$$

which is the usual equation of motion with gravitational potential Φ.