Linearized gravity

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1 Linearized gravity

When dealing with the Newtonian limit of General Relativity we have already met the weak-field condition, whereby the metric tensor is approximately equal to $\eta_{\mu\nu}$ but for a small perturbation $h_{\mu\nu}$ (such that $|h_{\mu\nu}| \ll 1$)

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

and the inverse metric tensor is

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

These definitions imply

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right) \tag{3}$$

$$\approx \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta} \right). \tag{4}$$

then

$$R^{\mu}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta}\Gamma^{\mu}_{\delta\gamma} \approx \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} \tag{5}$$

$$\approx \frac{1}{2} \eta^{\mu\nu} \left[\left(\partial_\alpha \partial_\beta h_{\nu\gamma} + \partial_\beta \partial_\gamma h_{\nu\alpha} - \partial_\beta \partial_\nu h_{\alpha\gamma} \right) - \left(\partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\beta \partial_\gamma h_{\nu\alpha} - \partial_\nu \partial_\gamma h_{\alpha\beta} \right) \right] \tag{6}$$

$$= \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h_{\nu\gamma} - \partial_\beta \partial_\nu h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\nu \partial_\gamma h_{\alpha\beta} \right) \tag{7}$$

Correspondingly, we find a linearized expression for the Ricci tensor

$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} \approx \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h_{\nu\mu} - \partial_\beta \partial_\nu h_{\alpha\mu} - \partial_\alpha \partial_\mu h_{\nu\beta} + \partial_\nu \partial_\mu h_{\alpha\beta} \right) \tag{8}$$

$$= \frac{1}{2} \left(\partial_\alpha \partial_\beta h^{\mu}_{\mu} - \partial_\beta \partial^\mu h_{\alpha\mu} - \partial_\alpha \partial^\mu h_{\mu\beta} + \partial^\mu \partial_\mu h_{\alpha\beta} \right) \tag{9}$$

$$= \frac{1}{2} \left(\Box^2 h_{\alpha\beta} + \partial_\alpha \partial_\beta h - \partial_\beta \partial^\mu h_{\alpha\mu} - \partial_\alpha \partial^\mu h_{\mu\beta} \right) \tag{10}$$

where $h = h^{\mu}_{\mu}$ and where

$$\Box^2 = \partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

is the d'Alambertian operator. From this it follows that the Ricci scalar is

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = \Box^2 h - \partial_\mu \partial_\nu h^{\mu\nu}, \tag{11}$$

and finally we obtain Einstein's tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}\left(\Box^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\right), \tag{12}$$

and the Einstein equation for linearized gravity

$$G_{\mu\nu} = \frac{1}{2}\left(\Box^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\right) = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{13}$$

Eq. (13) becomes slightly simpler by redefining the perturbation variables:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

(*trace-reversed perturbation variables*), such that

$$\bar{h} = \bar{h}^\mu_\mu = h - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h = h - \frac{1}{2}\delta^\mu_\mu h = h - 2h = -h$$

and therefore

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}.$$

Using the trace-reversed perturbation variables in eq (13), we find

$$\begin{aligned} &\Box^2\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box^2\bar{h} - \partial_\mu\partial_\nu\bar{h} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} + \frac{1}{2}\partial_\mu\partial_\nu\bar{h} \\ &\qquad - \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} + \frac{1}{2}\partial_\nu\partial_\mu\bar{h} + \eta_{\mu\nu}\Box^2\bar{h} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\Box^2\bar{h} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \end{aligned} \tag{14}$$

i.e.,

$$\Box^2\bar{h}_{\mu\nu} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} - \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{15}$$

which is the Einstein equation for linearized gravity with trace-reversed perturbation variables.

2 Gauge transformations

We have already met the ambiguities associated with the freedom of choice of the coordinate system, and we easily guess that fluctuations of the metric tensor can be due to

1. perturbations of space-time
2. perturbations of the coordinate system
3. both 1. and 2.

We can understand the effect of coordinate perturbations by making small *gauge transformation.* Consider two coordinate systems which differ by a small translation ξ^μ:

$$x'^\mu = x^\mu + \xi^\mu; \quad x^\mu = x'^\mu - \xi^\mu \quad (|\xi^\mu| \ll 1)$$

so that the coordinate transformation matrices are

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu \tag{16}$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial'_\nu \xi^\mu = \delta^\mu_\nu - \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial \xi^\mu}{\partial x^\alpha} = \delta^\mu_\nu - (\delta^\alpha_\nu - \partial'_\nu \xi^\alpha)\frac{\partial \xi^\mu}{\partial x^\alpha} \approx \delta^\mu_\nu - \partial_\nu \xi^\mu. \tag{17}$$

In particular, the metric tensor transforms as follows:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta} \tag{18}$$

$$\approx g_{\mu\nu} - \partial_\nu \xi^\beta \, g_{\mu\beta} - \partial_\mu \xi^\alpha \, g_{\alpha\nu} \tag{19}$$

$$= g_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \tag{20}$$

For this small transformation, we have, again,

$$g_{\mu\nu} \approx n_{\mu\nu} + h_{\mu\nu},$$

therefore, when expressed in terms of perturbation variables, the transformation eq. (20) becomes

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \tag{21}$$

We find the transformation of the trace-reversed perturbation variables evaluating first the transformed trace variable

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\partial_\nu \xi_\mu - \eta^{\mu\nu}\partial_\mu \xi_\nu = h - 2\partial_\mu \xi^\mu, \tag{22}$$

then

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\partial_\alpha \xi^\alpha\right) \tag{23}$$

$$= \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha \tag{24}$$

As they should, these coordinate transformations do not affect (at this order) the Riemann tensor:

$$\begin{aligned}
R'^{\mu}_{\alpha\beta\gamma} &= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha\partial_\beta h'_{\nu\gamma} - \partial_\beta\partial_\nu h'_{\alpha\gamma} - \partial_\alpha\partial_\gamma h'_{\nu\beta} + \partial_\nu\partial_\gamma h'_{\alpha\beta}\right) &(25)\\
&= \frac{1}{2}\eta^{\mu\nu}\left[\partial_\alpha\partial_\beta(h_{\nu\gamma} - \partial_\nu\xi_\gamma - \partial_\gamma\xi_\nu) - \partial_\beta\partial_\nu(h_{\alpha\gamma} - \partial_\alpha\xi_\gamma - \partial_\gamma\xi_\alpha)\right.\\
&\quad \left. -\partial_\alpha\partial_\gamma(h_{\nu\beta} - \partial_\nu\xi_\beta - \partial_\beta\xi_\nu) + \partial_\nu\partial_\gamma(h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha)\right] &(26)\\
&= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha\partial_\beta h_{\nu\gamma} - \partial_\beta\partial_\nu h_{\alpha\gamma} - \partial_\alpha\partial_\gamma h_{\nu\beta} + \partial_\nu\partial_\gamma h_{\alpha\beta}\right) = R^{\mu}_{\alpha\beta\gamma} &(27)
\end{aligned}$$

Going back to eq. (15)

$$\Box^2\bar{h}_{\mu\nu} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} - \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\mu\nu},$$

we see that we can fix the coordinate system in a particularly advantageous way if we set

$$\partial_\nu\bar{h}^{\mu\nu} = 0 \tag{28}$$

(the *Lorenz gauge*), then it is straightforward to see that the equation reduces to the system

$$\Box^2\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{29}$$

$$\partial_\nu\bar{h}^{\mu\nu} = 0 \tag{30}$$

2.1 Existence of the Lorenz gauge

Is there a coordinate system that actually satisfies the Lorenz condition? If it exists, then there must be a coordinate transformation

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha \tag{31}$$

such that $\partial_\nu\bar{h}'^{\mu\nu} = 0$. Indeed, using eq. (31), we find

$$\begin{aligned}
0 = \partial^\nu\bar{h}'_{\mu\nu} &= \partial^\nu\bar{h}_{\mu\nu} - \partial^\nu\partial_\nu\xi_\mu - \partial^\nu\partial_\mu\xi_\nu + \eta_{\mu\nu}\partial^\nu\partial_\alpha\xi^\alpha &(32)\\
&= \partial^\nu\bar{h}_{\mu\nu} - \Box^2\xi_\mu - \partial_\mu\partial^\nu\xi_\nu + \partial_\mu\partial^\nu\xi_\nu &(33)\\
&= \partial^\nu\bar{h}_{\mu\nu} - \Box^2\xi_\mu &(34)
\end{aligned}$$

We end up with the equation

$$\partial^\nu\bar{h}_{\mu\nu} = \Box^2\xi_\mu \tag{35}$$

which is a wave equation with a source term. It turns out that – under very general assumptions – it can always be solved as a by functions of the form $g(x) + g_0(x)$, where g is a particular solution which takes into account the source term on the r.h.s. of the equation, and g_0 is the general solution of the associated homogeneous equation

$$\partial^\nu\bar{h}_{\mu\nu} = 0, \tag{36}$$

therefore we conclude that we can *always* find a suitable transformation that takes us to a coordinate system where the Lorenz gauge holds.