Equivalence of the geodesic equations

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The geodesic equations found with the method of parallel transport of the tangent vector are:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{1}$$

and the geodesic equations found from the relativistic Lagrangian of a free particle are

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0; \tag{2}$$

Here we show that they are equivalent.

Notice first that

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^\mu}{d\tau}\right) = \frac{dg_{\alpha\mu}}{d\tau}\frac{dx^\mu}{d\tau} + g_{\alpha\mu}\frac{dx^\mu}{d\tau} \tag{3}$$

$$= \partial_\nu g_{\alpha\mu}\dot{x}^\mu\dot{x}^\nu + g_{\alpha\mu}\frac{d^2x^\mu}{d\tau^2}, \tag{4}$$

therefore

$$\partial_\nu g_{\alpha\mu}\dot{x}^\mu\dot{x}^\nu + g_{\alpha\mu}\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{5}$$

Next, we exchange μ and α

$$\partial_\nu g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\nu + g_{\alpha\mu}\frac{d^2x^\alpha}{d\tau^2} - \frac{1}{2}\partial_\mu g_{\alpha\nu}\dot{x}^\alpha\dot{x}^\nu = 0, \tag{6}$$

rearrange the equation

$$\frac{d^2x^\beta}{d\tau^2} + \frac{1}{2}g^{\beta\mu}\left(\partial_\nu g_{\alpha\mu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu}\right)\dot{x}^\alpha\dot{x}^\nu = 0, \tag{7}$$

exchange β and α

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}g^{\alpha\mu}\left(\partial_\nu g_{\beta\mu} + \partial_\nu g_{\beta\mu} - \partial_\mu g_{\beta\nu}\right)\dot{x}^\beta\dot{x}^\nu = 0, \tag{8}$$

and finally, β and μ

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}g^{\alpha\beta}\left(\partial_\nu g_{\beta\mu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu = 0 \tag{9}$$

Expanding the parenthesis, the second term is $\partial_\nu g_{\beta\mu}\dot{x}^\mu\dot{x}^\nu = \partial_\mu g_{\beta\nu}\dot{x}^\mu\dot{x}^\nu$ (exchanging μ and ν), and therefore

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}g^{\alpha\beta}\left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu = 0, \tag{10}$$

and recalling that

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell k}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell} \right)$$

we find

$$\frac{d^2 x^\alpha}{d\tau^2} + \frac{1}{2} g^{\alpha\beta} \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \tag{11}$$

and conclude the proof of the equivalence.