# Geodesic deviation

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We start from the geodesic equations

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0, \tag{1}$$

where the $u$ is the curve parameter and $x^a = x^a(u)$ (latin indexes, because these equations hold true for any Riemann manifold). Next, we consider two neighboring geodesics and the identified by the coordinates $x^a(u)$ and $\tilde{x}^a(u)$ and their (small) difference – the *geodesic deviation* – at given $u$, $\xi^a(u) = \tilde{x}^a(u) - x^a(u)$. When we consider the geodesic equation for $\tilde{x}^a(u)$

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} = 0, \tag{2}$$

where $\tilde{\Gamma}^a_{bc}$ is evaluated at $\tilde{x}^a(u) = x^a(u) + \xi^a(u)$, and subtract eq. (1) from eq. (2), we find

$$\frac{d^2\xi^a}{du^2} = \frac{d^2\tilde{x}^a}{du^2} - \frac{d^2x^a}{du^2} = -\left(\tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du}\right). \tag{3}$$

We can expand the Christoffel symbol $\tilde{\Gamma}^a_{bc}$ to first order in $\xi^a$ as follows:

$$\tilde{\Gamma}^a_{bc} \approx \Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\xi^d, \tag{4}$$

then, substituting in eq. (3) and neglecting terms that are superlinear in $\xi$ and its derivatives, we find

$$\frac{d^2\xi^a}{du^2} = -\left(\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\,\xi^d\right)\frac{d(x^b+\xi^b)}{du}\frac{d(x^c+\xi^c)}{du} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} \tag{5}$$

$$\approx -\left(\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\,\xi^d\right)\left[\frac{dx^b}{du}\frac{dx^c}{du} + \dot{\xi}^b\frac{dx^c}{du} + \dot{\xi}^c\frac{dx^b}{du}\right] + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} \tag{6}$$

$$\approx -\Gamma^a_{bc}\dot{\xi}^b\frac{dx^c}{du} - \Gamma^a_{bc}\dot{\xi}^c\frac{dx^b}{du} - \partial_d\Gamma^a_{bc}\,\xi^d\frac{dx^b}{du}\frac{dx^c}{du} \tag{7}$$

The last equation can be rearranged as follows:

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d - \Gamma^a_{bc}\xi^b\ddot{x}^c + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{8}$$

Now notice that the geodesic equation (1) can be rewritten in the form

$$\ddot{x}^c = -\Gamma^c_{de}\dot{x}^d\dot{x}^e \tag{9}$$

which holds on the geodesic: after substitution of the second derivative $\ddot{x}^c$, eq. (8) becomes

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{10}$$

We recall here the definition of the absolute derivative

$$\frac{Dv^a}{du} = \dot{v}^a + \Gamma^a_{bc}v^b\dot{x}^c \tag{11}$$

and we remark that the term enclosed in the parenthesis is the absolute derivative of $\xi$. We also note that

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) = \frac{D}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e \tag{12}$$

$$= \frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e. \tag{13}$$

With this rearrangement of the first term, eq. (10) becomes

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0, \tag{14}$$

or also

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\dot{\xi}^d\dot{x}^e - \Gamma^a_{de}\Gamma^d_{bc}\xi^b\dot{x}^c\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{15}$$

Taking into account the symmetry of the lower indexes of the Christoffel symbols and with the proper substitutions of indexes in the contractions, we see that the second and the sixth term in eq. (15) have the same value and opposite sign, and eq. (15) simplifies to

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\Gamma^d_{bc}\xi^b\dot{x}^c\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0, \tag{16}$$

and after rearranging the terms and changing the names of indices in contractions

$$\frac{D^2\xi^a}{du^2} + \left(\partial_b\Gamma^a_{de}\,\xi^b\dot{x}^d\dot{x}^e - \partial_d\Gamma^a_{be}\,\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e - \Gamma^a_{ce}\Gamma^c_{bd}\xi^b\dot{x}^d\dot{x}^e\right) = 0, \tag{17}$$

i.e.,

$$\frac{D^2\xi^a}{du^2} + \left(\partial_b\Gamma^a_{cd} - \partial_c\Gamma^a_{bd} + \Gamma^a_{be}\Gamma^e_{cd} - \Gamma^a_{ce}\Gamma^e_{bd}\right)\xi^b\dot{x}^c\dot{x}^d = 0. \tag{18}$$

Recalling the definition of the Riemann curvature tensor

$$R^d_{abc} = \partial_b\Gamma^d_{ac} - \partial_c\Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}, \tag{19}$$

we see that eq. (18) reduces to

$$\frac{D^2\xi^a}{du^2} + R^a_{cbd}\xi^b\dot{x}^c\dot{x}^d = 0. \tag{20}$$

the *equation of geodesic deviation.*