Total emitted GW power

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Earlier in the course, we found that the generic amplitude of the GW strain in the TT gauge, with propagation in the z direction is

$$[A^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

for a gravitational wave with strain

$$h^{\mu\nu} = A^{\mu\nu} \cos(\omega t - kz) \tag{2}$$

Clearly, taking a generic matrix with propagation in a generic direction, we still have vanishing time components, and the matrix is

$$[A^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & A^{13} \\ 0 & A^{21} & A^{22} & A^{23} \\ 0 & A^{31} & A^{32} & A^{33} \end{pmatrix} \tag{3}$$

Now we have to project it in a plane perpendicular to the direction of propagation and make it traceless. We start with the projection operator of a vector in the direction perpendicular to a unit vector $\mathbf{n}$:

$$P^j_k = \delta^j_k - n^j n_k. \tag{4}$$

A true projector operator should be such that $P^j_m P^m_k = P^j_k$: indeed, we find

$$P^j_m P^m_k = (\delta^j_m - n^j n_m)(\delta^m_k - n^m n_k) = \delta^j_k - n^j n_k - n^j n_k + (n_m n^m) n^j n_k = \delta^j_k - n^j n_k = P^j_k \tag{5}$$

In the case of $\mathbf{n} = \hat{\mathbf{z}}$ it is easy to see that

$$[P^j_k] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Since a matrix like A^{ij} transforms like the tensor product of two contravariant vectors, the projected (transverse) matrix is

$$[A^{jk}_T] = [P^j_m A^{mn} P^k_n] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{11} & A^{12} & 0 \\ A^{21} & A^{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

as we easily guess.

We make the matrix traceless by subtracting the trace equally from each diagonal element:

$$[A^{jk}_{TT}] = \begin{pmatrix} A^{xx} & A^{xy} & 0 \\ A^{yx} & A^{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}(A^{xx} + A^{yy}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(A^{xx} - A^{yy}) & A^{xy} & 0 \\ A^{yx} & -\frac{1}{2}(A^{xx} - A^{yy}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8)$$

Formally, the trace of the transverse matrix is

$$\eta_{ab} A^{ab}_{T} = \eta_{ab} P^a_m P^b_n A^{mn} = P_{bm} P^b_n A^{mn} = P_{mn} A^{mn}, \quad (9)$$

this means that, in general, the transverse-traceless matrix is given by the following expression

$$A^{ij}_{TT} = \left(P^i_m P^j_n - \frac{1}{2} P^{ij} P_{mn} \right) A^{mn} \quad (10)$$

We use these operators and the equation for the energy flux that we discussed earlier

$$\text{energy flux} = \frac{G}{8\pi c^5} \frac{\left\langle \dddot{I\!\!\!-}_{ij} \dddot{I\!\!\!-}^{ij} \right\rangle}{r^2}, \quad (11)$$

to find the flux in a generic direction. Thus,

$$\dddot{I\!\!\!-}^{TT}_{ij} \dddot{I\!\!\!-}^{ij}_{TT} = \left(P^i_m P^j_n - \frac{1}{2} P^{ij} P_{mn} \right) \dddot{I\!\!\!-}^{mn} \left(P^k_i P^\ell_j - \frac{1}{2} P_{ij} P^{k\ell} \right) \dddot{I\!\!\!-}_{k\ell} \quad (12)$$

$$= \left(P^k_m P^\ell_n - \frac{1}{2} P^{k\ell} P_{mn} - \frac{1}{2} P^{k\ell} P_{mn} + \frac{1}{2} P^{k\ell} P_{mn} \right) \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} \quad (13)$$

$$= \left(P^k_m P^\ell_n - \frac{1}{2} P^{k\ell} P_{mn} \right) \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} \quad (14)$$

Now, notice that

$$P_{mn} \dddot{I\!\!\!-}^{mn} = P^m_n \dddot{I\!\!\!-}^n_m = (\delta^m_n - n^m n_n) \dddot{I\!\!\!-}^n_m = \dddot{I\!\!\!-}^n_n - n^m n_n \dddot{I\!\!\!-}^n_m = -n_m n_n \dddot{I\!\!\!-}^{mn} \quad (15)$$

therefore the last term in expression (14) is

$$-\frac{1}{2} P^{k\ell} P_{mn} \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} = -\frac{1}{2} n_k n_\ell n_m n_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}^{k\ell}. \quad (16)$$

The first term in expression (14) can be expanded as follows:

$$P^k_m P^\ell_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} = \left(\delta^k_m - n^k n_m\right) \left(\delta^\ell_n - n^\ell n_n\right) \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} \quad (17)$$

$$= \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{mn} - n^k n_m \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{kn} - n^\ell n_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{m\ell} + n^k n_m n^\ell n_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{k\ell} \quad (18)$$

$$= \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{mn} - n^k n_m \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{kn} - n^k n_m \dddot{I\!\!\!-}^{nm} \dddot{I\!\!\!-}_{nk} + n_k n_m n_\ell n_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}^{k\ell} \quad (19)$$

$$= \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{mn} - 2 n^k n_m \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{kn} + n_k n_m n_\ell n_n \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}^{k\ell} \quad (20)$$

Finally, combining all the pieces, we find

$$\dddot{I}^{TT}_{ij}\,\dddot{I}^{ij}_{TT} = \dddot{I}^{mn}\,\dddot{I}_{mn} - 2n_k n_m\,\dddot{I}^{mn}\,\dddot{I}^{k}_{n} + \frac{1}{2}n_k n_m n_\ell n_n\,\dddot{I}^{mn}\,\dddot{I}^{k\ell} \tag{21}$$

and the equation for the energy flux in direction $\mathbf{n}$ is

$$\begin{aligned}\text{energy flux} &= \frac{G}{8\pi c^5}\frac{\left\langle \dddot{I}^{TT}_{ij}\,\dddot{I}^{ij}_{TT}\right\rangle}{r^2}\\ &= \frac{G}{16\pi c^5 r^2}\left\langle 2\,\dddot{I}^{mn}\,\dddot{I}_{mn} - 4n^k n^m\,\dddot{I}^{n}_{m}\,\dddot{I}_{nk} + n^k n^m n^\ell n^n\,\dddot{I}_{mn}\,\dddot{I}_{k\ell}\right\rangle \end{aligned} \tag{22}$$

Now we can integrate this energy flux (irradiance) of the gravitational wave over all direction to find the total emitted power. To this end we need an explicit expression for the unit vector $\mathbf{n}$. We take the z-axis as a reference and write

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

so that, by integrating over the surface of a sphere of radius r, the total emitted power is

$$P_{\mathrm{GW}} = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta\,\frac{G}{8\pi c^5}\left\langle \dddot{I}^{TT}_{ij}\,\dddot{I}^{ij}_{TT}\right\rangle \tag{23}$$

$$= \frac{G}{16\pi c^5}\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta\left(2\left\langle \dddot{I}^{mn}\,\dddot{I}_{mn}\right\rangle - 4n^k n^m\left\langle \dddot{I}^{n}_{m}\,\dddot{I}_{nk}\right\rangle + n^k n^m n^\ell n^n\left\langle \dddot{I}_{mn}\,\dddot{I}_{k\ell}\right\rangle\right) \tag{24}$$

The last formula splits into separate integrals; the **first integral** is easy to evaluate

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta = 4\pi. \tag{25}$$

The **second integral** is

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta\; n^k n^m \tag{26}$$

and we can distinguish different cases:

1. $k \neq m$: in practice, because of the symmetry with respect to index exchanges, this corresponds to three products, $n^x n^y$, $n^x n^z$, and $n^y n^z$, with the corresponding integrals evaluated below

   (a)

$$\begin{aligned}\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta\; n^x n^y &= \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta\;\sin^2\theta\cos\phi\sin\phi\\ &= \frac{1}{2}\int_{-1}^{+1}(1-\cos^2\theta)d\cos\theta\int_0^{2\pi}\sin 2\phi\; d\phi = 0\end{aligned} \tag{27}$$

(b)

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^x n^z = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \;\; \sin\theta\cos\theta\cos\phi = 0 \tag{28}$$

(c)

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^y n^z = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \;\; \sin\theta\cos\theta\sin\phi = 0 \tag{29}$$

2. $k = m$: this corresponds to three products, $n^x n^x$, $n^y n^y$, and $n^z n^z$, however we expect them to be equal because of the spherical symmetry of the problem (arbitrary choice of the reference axes for vector $\mathbf{n}$), and we only need to evaluate the simplest one

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^z n^z = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \;\; \cos^2\theta = \frac{4\pi}{3} \tag{30}$$

Finally, we can summarize these result with the single formula

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^i n^j = \frac{4\pi}{3}\eta^{ij} \tag{31}$$

The **third integral** is

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^k n^m n^\ell n^n \tag{32}$$

In the previous calculation we have seen that only the terms with an even number of equal factors survive: the reason is that those with an odd number are also odd functions with respect to even integration intervals. For the same reason, here only the terms which contain two pairs of equal indices, or where all indices are equal, survive. Therefore we must consider the terms $n^x n^x n^y n^y$, $n^x n^x n^z n^z$, and $n^y n^y n^z n^z$ (pairs), and the terms $n^x n^x n^x n^x$, $n^y n^y n^y n^y$, and $n^z n^z n^z n^z$:

1. $n^x n^x n^y n^y$ (and similar terms): the naming of the pairs does not really matter, because of the arbitrariness in choosing the reference frame, therefore we only need to evaluate one integral, with integrand $n^x n^x n^z n^z$

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^x n^x n^z n^z = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \;\; \sin^2\theta\cos^2\theta\cos^2\phi \tag{33}$$

$$= \int_0^{2\pi} \cos^2\phi \; d\phi \int_{-1}^{+1} dx(x^2 - x^4) = \frac{4\pi}{15} \tag{34}$$

2. $n^x n^x n^x n^x$ (and similar terms): in this case all the three integrals must be the same, we only take the integrand $n^z n^z n^z n^z$

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \; n^z n^z n^z n^z = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \;\; \cos^4\theta = \frac{4\pi}{5} \tag{35}$$

We can take into account all the index combinations that produce nonvanishing values of the integral with the sum

$$\frac{4\pi}{15}\left(\eta^{km}\eta^{\ell n}+\eta^{k\ell}\eta^{mn}+\eta^{kn}\eta^{m\ell}\right)$$

and indeed, each element of the sum selects just one of the combinations that include unequal pairs, contributing with $4\pi/15$, while in the case of all indices equal, all terms contribute, and therefore the corresponding value is $4\pi/5$ as it should be.

Thus, the GW power emitted by the source is

$$P_{\rm GW} = \frac{G}{16\pi c^5}\left[8\pi\left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle - \frac{16\pi}{3}\eta^{km}\left\langle \dddot{I}_m^{\ n}\dddot{I}_{nk}\right\rangle + \frac{4\pi}{15}\left(\eta^{km}\eta^{\ell n}+\eta^{k\ell}\eta^{mn}+\eta^{kn}\eta^{m\ell}\right)\left\langle \dddot{I}_{mn}\dddot{I}_{k\ell}\right\rangle\right] \tag{36}$$

$$= \frac{G}{4c^5}\left[2\left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle - \frac{4}{3}\left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle + \frac{1}{15}\left(\left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle + \left\langle \dddot{I}_m^{\ m}\dddot{I}_n^{\ n}\right\rangle + \left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle\right)\right] \tag{37}$$

$$= \frac{G}{5c^5}\left\langle \dddot{I}^{mn}\dddot{I}_{mn}\right\rangle \tag{38}$$

(recalling that $\dddot{I}^{mn}$ is traceless).