GW heuristics

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We found earlier that when we define the spatial *quadrupole tensor* of the mass distribution

$$Q_{ij} = \int \rho x_i x_j \; d^3\mathbf{x} \tag{1}$$

then the amplitude of the gravitational waves emitted by a mass with density $\rho(t, \mathbf{x})$, is given by the *quadrupole formula*

$$\bar{h}_{ij} \approx -\frac{2G}{c^4 r} \frac{d^2 Q_{ij}}{dt^2} \tag{2}$$

where r is the distance to the source and t is the retarded time.

We also found that there is a very useful specialization of the Lorentz gauge, the TT gauge, a coordinate system that is comoving with the wave itself. In the TT gauge, free particles remain at constant coordinate locations, although their proper separations change. We find the amplitude of the GW in the TT gauge by projecting the quadrupole tensor in the plane perpendicular to the direction of the wave, and by removing the trace of the projected tensor.

It is easy to see that the quadrupole tensor of a spherical mass distribution is proportional to δ_{ij}, so that it becomes the null tensor when we remove the trace. This means that spherically symmetric motions do not produce any gravitational radiation.

Dimensionally, the components of the second derivative of the quadrupole tensor in eq. (2) have units [mass][speed]^2, therefore we expect the components of the strain to be bounded

$$h \lesssim \frac{2G}{c^4 r} (Mv^2)_{\text{nonsph}} \tag{3}$$

where $(Mv^2)_{\text{nonsph}}$ is twice the kinetic energy for the nonspherical part of the mass distribution. The bound can be written slightly differently using the gravitational potential

$$h < \frac{2\phi_{\text{ext}}}{c^4} (v^2)_{\text{nonsph}} \tag{4}$$

with the assumption that the entire mass is involved in the nonspherical motion, so that $(Mv^2)_{\text{nonsph}} = M(v^2)_{\text{nonsph}}$. Now, recall the virial theorem (see the Appendix for a proof) which states that in a gravitating system, $M\langle v^2 \rangle = \langle \phi_{\text{int}} \rangle$, where the average gravitational potential $\langle \phi_{\text{int}} \rangle$ is that of the internal interactions in the gravitating system that emits gravitational waves. Then, if ϕ_{int} is the maximum potential internal to the system

$$v^2_{\text{nonsph}} < \phi_{\text{int}} \tag{5}$$

and finally

$$h < \frac{2\phi_{\rm ext}\phi_{\rm int}}{c^4} \tag{6}$$

a bound which is attained if the system has no spherical symmetry, like a binary star system.

Finally, we note that nonspherical part of the quadrupole tensor is given by the *reduced quadrupole tensor*

$$\bar{I}^{ij} = \int_{\rm source} \rho \left(x^i x^j - \frac{1}{3}\eta^{ij} r^2 \right) d^3\mathbf{x}, \tag{7}$$

so that equation (2) becomes

$$\bar{h}_{ij} \approx -\frac{2G}{c^4 r}\,\frac{d^2 \bar{I}_{ij}}{dt^2} \tag{8}$$

Appendix: the virial theorem

Consider the quantity $\lambda = p_i q_i$ (assuming as usual the Einstein summation convention), then, from Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

we find

$$\frac{d\lambda}{dt} = p_i \frac{dq_i}{dt} + q_i \frac{dp_i}{dt} = p_i \frac{\partial H}{\partial p_i} - q_i \frac{\partial H}{\partial q_i}. \tag{9}$$

By integrating (9), we obtain

$$\frac{\lambda(\tau) - \lambda(0)}{\tau} = \frac{1}{\tau} \int_0^\tau \left(p_i \frac{\partial H}{\partial p_i} - q_i \frac{\partial H}{\partial q_i} \right) dt. \tag{10}$$

In a periodic system the l.h.s. of (10) vanishes over a period, and it vanishes anyway for bounded λ as $\tau \to \infty$. In both cases we have

$$\frac{1}{\tau} \int_0^\tau p_i \frac{\partial H}{\partial p_i} dt = \frac{1}{\tau} \int_0^\tau q_i \frac{\partial H}{\partial q_i} dt, \tag{11}$$

which writes as follow when we introduce the time average $\langle \ldots \rangle$:

$$\left\langle p_i \frac{\partial H}{\partial p_i} \right\rangle = \left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle. \tag{12}$$

Next, we introduce the *virial function*

$$\mathcal{V} = -q_i \frac{\partial H}{\partial q_i} = q_i \dot{p}_i$$

and eq. (12) becomes

$$\langle \mathcal{V} \rangle = -\left\langle p_i \frac{\partial H}{\partial p_i} \right\rangle \tag{13}$$

The virial function was introduced by Clausius in the context of the (many-particle) theory of gases, but here we use it for a single particle moving in a central potential $\phi(r)$, so that

$$\langle \mathcal{V} \rangle = \langle \mathbf{r} \cdot \dot{\mathbf{p}} \rangle = -\langle \mathbf{p} \cdot \dot{\mathbf{r}} \rangle \tag{14}$$

and recalling that $T = (\mathbf{p} \cdot \dot{\mathbf{r}})/2$ is the kinetic energy is, and that the force is derived from the central potential $\dot{\mathbf{p}} = -\nabla\phi$, we obtain

$$2\langle T \rangle = \langle \mathbf{r} \cdot \nabla\phi \rangle. \tag{15}$$

Assuming a central potential $\phi(r) = K/r^n$, we find

$$\nabla\phi = -\frac{Kn}{r^{n+2}} \mathbf{r} = -n \frac{\phi}{r} \hat{\mathbf{r}} \tag{16}$$

and finally

$$2\langle T \rangle = -n \langle \phi \rangle. \tag{17}$$

By extending the average to include also an average over many particles, this result extends to a system of particles.