Generation of gravitational waves

Edoardo Milotti

January 12, 2023

Here we reconsider the wave equation with source:

$$\Box^2 \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}. \tag{1}$$
$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{2}$$

Formally, eq. (1) is just like the equations for the individual electromagnetic vector potential components $A^\mu = (\phi/c, \mathbf{A})$ in the Lorentz gauge, in particular for the 0 component (the electric potential) $\phi = cA^0$ the equation is:

$$\Box^2 \phi = \frac{\rho}{\varepsilon_0} \tag{3}$$
$$\partial_\mu A^\mu = 0 \tag{4}$$

The solution of eq. (3) in vacuum, based on retarded potentials, is well-known

$$\phi(ct, \mathbf{x}) = \int \frac{\rho(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \tag{5}$$

where $\mathbf{r}_0$ is the position of a small volume of the source, $\mathbf{r}$ is the position where the field is determined, and $r = |\mathbf{r} - \mathbf{r}_0|$ is their distance. Since, formally, eq. (1) can be obtained from eq. (3) with the substitution $1/\varepsilon_0 \to -16\pi G/c^4$, we see that the solution of eq. (1) is

$$\bar{h}^{\mu\nu}(ct, \mathbf{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \tag{6}$$

Under the following conditions:

- source size $\ll$ wavelength λ of the wave $\ll$ distance r to the source;
- $|\bar{h}^{\mu\nu}| \ll 1$;
- source is slow (all its parts move with speed $\ll c$);

the solution (6) approximates to

$$\bar{h}^{\mu\nu}(ct, \mathbf{x}) \approx -\frac{4G}{c^4 r} \int T^{\mu\nu}(ct - r, \mathbf{x}') d^3\mathbf{x}'. \tag{7}$$

Now notice that applying the Lorentz condition to both sides of eq. (1), we find

$$\Box^2 (\partial_\nu \bar{h}^{\mu\nu}) = -\frac{16\pi G}{c^4} \partial_\nu T^{\mu\nu} = 0, \tag{8}$$

i.e.,

$$\partial_\nu T^{\mu\nu} = 0 \tag{9}$$

where the time part is

$$\partial_\nu T^{0\nu} = \partial_0 T^{00} + \partial_k T^{0k} = 0, \tag{10}$$

and the space part is

$$\partial_\nu T^{i\nu} = \partial_0 T^{i0} + \partial_k T^{ik} = 0; \tag{11}$$

we use these equations to prove an identity that helps evaluating the integral (7).

We start with

$$\int \partial_k (T^{ik} x^j) \, d^3\mathbf{x} = \int \left[\partial_k T^{ik} x^j + T^{ik} \delta^j_k \right] d^3\mathbf{x} = \int \left[-\partial_0 T^{i0} x^j + T^{ij} \right] d^3\mathbf{x} \tag{12}$$

where the integral on the l.h.s. is a divergence, and thanks to Gauss' theorem it is equivalent to a surface integral on the boundary of the mass distribution; however, at the boundary, T^{ik} drops to zero, and the whole integral evaluates to zero. Therefore, using this result and the small-speed approximation,

$$\int T^{ij} \, d^3\mathbf{x} = \int \partial_0 T^{i0} x^j \, d^3\mathbf{x} \approx \frac{1}{c} \frac{d}{dt} \int T^{i0} x^j \, d^3\mathbf{x} \tag{13}$$

and exchanging indexes and summing, we find

$$\int T^{ij} \, d^3\mathbf{x} \approx \frac{1}{2c} \frac{d}{dt} \int \left(T^{i0} x^j + T^{j0} x^i \right) d^3\mathbf{x}. \tag{14}$$

Next, we remark that

$$\begin{aligned} \int \partial_k \left(T^{0k} x^i x^j \right) d^3\mathbf{x} &= \int \left(\partial_k T^{0k} x^i x^j + T^{0k} x^i \delta^j_k + T^{0k} x^j \delta^i_k \right) d^3\mathbf{x} \\ &= \int \partial_k T^{0k} x^i x^j \, d^3\mathbf{x} + \int \left(T^{0j} x^i + T^{0i} x^j \right) d^3\mathbf{x}, \end{aligned} \tag{15}$$

where the l.h.s. volume integral has an integrand which is a divergence and can be transformed into a surface integral that vanishes as above. Therefore,

$$\int \left(T^{0j} x^i + T^{0i} x^j \right) d^3\mathbf{x} = - \int \partial_k T^{0k} x^i x^j dV = \int \partial_0 T^{00} x^i x^j \, d^3\mathbf{x}, \tag{16}$$

and finally,

$$\int T^{ij} \, d^3\mathbf{x} \approx \frac{1}{2c^2} \frac{d^2}{dt^2} \int T^{00} x^i x^j \, d^3\mathbf{x}. \tag{17}$$

We can use this result and the slow-motion assumption $T^{00} \approx \rho c^2$, where ρ is the mass density, to obtain

$$\begin{aligned} \bar{h}^{ij}(ct, \mathbf{r}) &\approx -\frac{4G}{c^4 r} \int T^{ij}(ct - r, \mathbf{r}) \, d^3\mathbf{x} = -\frac{4G}{c^4 r} \frac{1}{2c^2} \frac{d^2}{dt^2} \left[\int T^{00} x^i x^j \, d^3\mathbf{x} \right]_{\text{retarded}} \\ &= -\frac{2G}{c^4 r} \frac{d^2}{dt^2} \left[\int \rho x^i x^j \, d^3\mathbf{x} \right]_{\text{retarded}}, \end{aligned} \tag{18}$$

where the integrals are evaluated at the retarded time. The integral

$$Q^{ij} = \int \rho x^i x^j \, d^3\mathbf{x}$$

is the *quadrupole tensor* of the mass distribution, so that the solution can also be written in the form

$$\bar{h}^{ij}(ct, \mathbf{x}) \approx -\frac{2G}{c^4 r} \, \ddot{Q}^{ij} \, (t - r/c) \tag{19}$$

GWs radiated by a rotating dumbbell

As an application of eq. (18), consider a dumbbell, with two masses M at the end of a massless rod of length $2R$, as illustrated in figure 1. The dumbbell rotates in the (x^1, x^2) plane with constant angular speed ω about its midpoint.

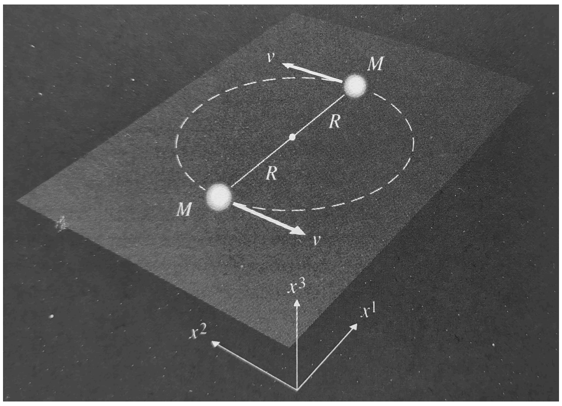

Figure 1: Example of a "dumbbell". In this case the stars in a binary system orbit one around the other in a circle of radius R (figure from S. M. Carroll, *Spacetime Geometry, an Introduction to General Relativity* Pearson, 2013).

Setting the origin of the coordinate system at the midpoint of the rod, the positions of the masses are

$$x^i = \pm(R\cos\omega t, R\sin\omega t, 0)$$

and we find

$$[\bar{h}^{ij}(ct, \mathbf{r})] = -\frac{4GMR^2}{c^4 r} \frac{d^2}{dt^2} \begin{pmatrix} \cos^2\omega t & \cos\omega t \sin\omega t & 0 \\ \cos\omega t \sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{retarded}} \tag{20}$$

$$= \frac{8GMR^2\omega^2}{c^4 r} \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{retarded}} \tag{21}$$

$$= \frac{8GMR^2\omega^2}{c^4 r} \begin{pmatrix} \cos 2\omega(t - r/c) & \sin 2\omega(t - r/c) & 0 \\ \sin 2\omega(t - r/c) & -\cos 2\omega(t - r/c) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{22}$$

This solution represents a gravitational wave with frequency 2ω. Both polarization components are equally represented and they are 90° out of phase, so that this is a circularly polarized gravitational wave.

It is easy to see that in the case of unequal masses M_1, M_2, and radii R_1, R_2, the previous result becomes

$$[\bar{h}^{ij}(ct, \mathbf{r})] = \frac{4G(M_1 R_1^2 + M_2 R_2^2)\omega^2}{c^4 r} \begin{pmatrix} \cos 2\omega(t - r/c) & \sin 2\omega(t - r/c) & 0 \\ \sin 2\omega(t - r/c) & -\cos 2\omega(t - r/c) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

We can use these results to find the order of magnitude of the amplitude of waves emitted by a laboratory-size apparatus. We assume here $M = 1$ kg, $R = 1$ m, $\omega = 1$ s^{-1}. Moreover, to satisfy the far-field approximation, $r \gg c/\omega$, then the amplitude of the gravitational waves is

$$h \approx \frac{8GMR^2\omega^2}{c^4 r} \ll \frac{8GMR^2\omega^3}{c^5} \sim 10^{-52},$$

which is totally undetectable with current technologies.

Further developments: GWs radiated by a rotating dumbbell

In the case of a binary star system with a large separation between stars (which the dumbbell with vanishing-mass rod approximates) and circular orbits, we can use the Keplerian formulas to obtain the frequency ω. With masses m_1 and m_2, with orbital radii r_1 and r_2, in the CM system the total momentum vanishes and we find

$$m_1 r_1 \omega = m_2 r_2 \omega \tag{24}$$

i.e.,

$$m_1 r_1 = m_2 r_2 \tag{25}$$

Moreover, the masses experience a centrifugal acceleration that must be balanced by the gravitational force, so that

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \tag{26}$$

The latter formula can be rearranged to obtain

$$r_1 \omega^2 = \frac{G m_2}{(r_1 + r_2)^2}; \quad r_2 \omega^2 = \frac{G m_1}{(r_1 + r_2)^2} \tag{27}$$

and therefore

$$\omega^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} \tag{28}$$

(a form of Kepler's third law). We shall develop these results further when discussing the radiation from compact binary systems.