Degrees of freedom of Einstein's equations

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October 23, 2022

Here we consider the degrees of freedom inside Einstein's equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\frac{8\pi G}{c^4} T^{\mu\nu}. \tag{1}$$

These equations fix the independent components of the metric tensor, however not all 10 of them, since we should still be free to transform to a different coordinate system with 4 coordinate transformation equations: $x'^\mu = f^\mu(\mathbf{x})$. This means that an equivalent of 4 Einstein's equation is automatically satisfied, so that we can only fix 6 independent quantities. To see that this is so, we must recall the Bianchi identity

$$\nabla_\sigma R_{\mu\alpha\beta\gamma} + \nabla_\beta R_{\mu\alpha\gamma\sigma} + \nabla_\gamma R_{\mu\alpha\sigma\beta} = 0 \tag{2}$$

and contract it to obtain a *contracted Bianchi identity.* We start with a double contraction with metric tensors and evaluate it

$$\begin{aligned} 0 &= g^{\alpha\beta} g^{\mu\gamma} \left(\nabla_\sigma R_{\mu\alpha\beta\gamma} + \nabla_\beta R_{\mu\alpha\gamma\sigma} + \nabla_\gamma R_{\mu\alpha\sigma\beta} \right) & (3)\\ &= \nabla_\sigma R - \nabla^\alpha R_{\alpha\sigma} - \nabla^\mu R_{\mu\sigma}, & (4) \end{aligned}$$

then we find the tensor equation

$$\nabla_\nu R = 2\nabla^\mu R_{\mu\nu} = 2\nabla_\mu R^\mu_\nu \tag{5}$$

Taking the covariant derivative of Einstein's tensor and using the symmetry of the Ricci tensor, we find in a LIF,

$$\nabla_\nu G^{\mu\nu} = \nabla_\nu R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \nabla_\nu R = \nabla_\nu R^{\mu\nu} - g^{\mu\nu} \nabla_\alpha R^\alpha_\nu = \nabla_\nu R^{\mu\nu} - \nabla_\alpha R^{\mu\alpha} = 0. \tag{6}$$

Thus we find that Einstein's tensor *automatically* satisfies the tensor equation $\nabla_\nu G^{\mu\nu} = 0$. We knew that this was necessary to satisfy $\nabla_\nu T^{\mu\nu} = 0$ and therefore the conservation of energy, and this was an important clue used by Einstein to finally derive his equations.

However, we could just as well have reasoned backwards, stating the need of having $\nabla_\nu G^{\mu\nu} = 0$ to ensure coordinate independence: then, the stress-energy tensor should satisfy $\nabla_\nu T^{\mu\nu} = 0$ as well, and energy-momentum conservation would follow as a consequence of the need of coordinate independence.