Compact Binary Coalescences

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August 1, 2023

To begin with, we partly repeat the calculation we already did to find the strain produced by a rotating dumbbell, and apply the formula for the total emitted power. As before, we consider a pair of stars with equal mass m in circular orbit about their common CM with orbit radius r, as illustrated in figure 1 (note however, that in the main text we use m instead of M and r instead of R). In this simple scheme, the stars rotate in the (x^1, x^2) plane with constant angular speed ω.

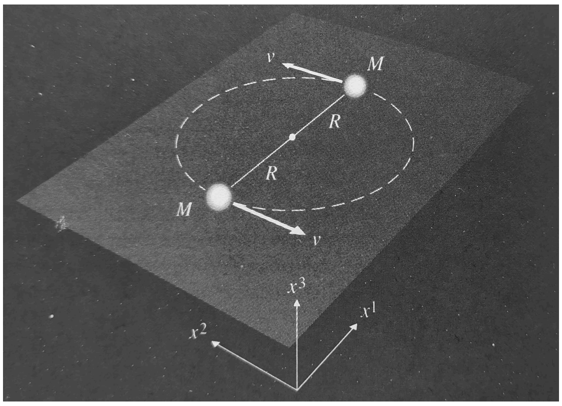

Figure 1: A binary star system. In this case the stars in the binary system orbit one around the other in a circle of radius R (figure from S. M. Carroll, *Spacetime Geometry, an Introduction to General Relativity* Pearson, 2013).

Setting the origin of the coordinate system at the CM, the positions of the masses are

$$x^i = \pm(r\cos\omega t, r\sin\omega t, 0)$$

and we find the quadrupole tensor

$$[Q^{ij}] = 2mr^2 \begin{pmatrix} \cos^2\omega t & \cos\omega t\sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

and the reduced quadrupole tensor

$$[{\rlap{-}I}^{ij}] = 2mr^2 \begin{pmatrix} \cos^2\omega t - 1/3 & \cos\omega t\sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t - 1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \tag{2}$$

Then

$$[\dddot{I\!\!\!-}^{ij}] = 2mr^2 \begin{pmatrix} 4\omega^3 \sin 2\omega t & -4\omega^3 \cos 2\omega t & 0 \\ -4\omega^3 \cos 2\omega t & -4\omega^3 \sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8\omega^3 mr^2 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

Therefore,

$$\left\langle \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{mn} \right\rangle = 128\, m^2 r^4 \omega^6, \tag{4}$$

and, finally, we find that this system emits energy as GWs at the rate

$$P_{\rm GW} = \frac{G}{5c^5} \left\langle \dddot{I\!\!\!-}^{mn} \dddot{I\!\!\!-}_{mn} \right\rangle = \frac{128G}{5c^5} m^2 r^4 \omega^6 \tag{5}$$

It can be shown (do it as an exercise), that for unequal masses m_1 and m_2, this equation can be written in the form

$$P_{\rm GW} = \frac{32G}{5c^5} \frac{m_1^2 m_2^2}{(m_1 + m_2)^2} (r_1 + r_2)^4 \omega^6 \tag{6}$$

Newtonian model of the inspiral phase in a compact binary system

We are now interested in the fate of a close pair of compact objects: as the emit energy in the form of GWs, they do this at the expense of potential energy and are gradually closer and closer, speeding up to higher and higher angular frequencies. The initial smooth and stable rotation eventually becomes a frantic race until the two object coalesce into one. This process is described in a sketch drawn many years ago by Kip Thorn (figure 2).

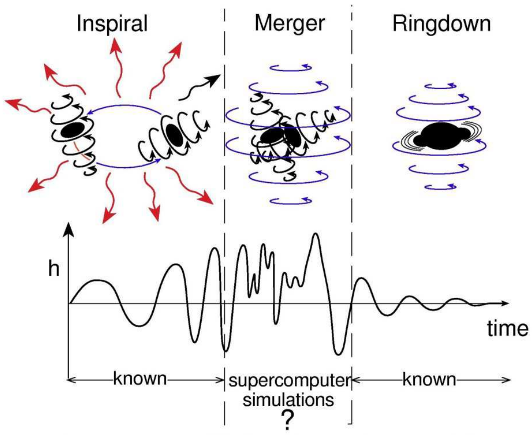

Figure 2: Drawing by Kip Thorne that illustrates the main phases of a compact binary coalescence (CBC). Initially the pair rotates at a nearly constant frequency, but at a later phase the energy loss due to GW emission is very fast, the frequency increases faster and faster while the object rotate closer and closer (inspiral phase). This continues until the objects merger. Finally, the remnant rings much like a bell with damped oscillations (ringdown).

We consider a non-relativistic two-body system with circular orbits about the center-of-mass (CM). The masses are m_1 and m_2, with orbital radii r_1 and r_2. Then, in the CM system the total momentum vanishes, and therefore

$$m_1 r_1 \omega = m_2 r_2 \omega \tag{7}$$

i.e.,

$$m_1 r_1 = m_2 r_2 \tag{8}$$

The masses also experience a centrifugal acceleration that must be balanced by the gravitational force, so that

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \tag{9}$$

The latter formula can be rearranged to obtain

$$r_1 \omega^2 = \frac{G m_2}{(r_1 + r_2)^2}; \quad r_2 \omega^2 = \frac{G m_1}{(r_1 + r_2)^2} \tag{10}$$

and therefore

$$\omega^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} \tag{11}$$

(a form of Kepler's third law), so that we can express the sum of radii as a function of total mass and frequency

$$r_1 + r_2 = \left(\frac{G(m_1 + m_2)}{\omega^2} \right)^{1/3} \tag{12}$$

Finally, the total energy of the system is

$$E_{\text{tot}} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 - \frac{G m_1 m_2}{(r_1 + r_2)} \tag{13}$$

Now notice that the moment of inertia of the system about the CM is, using eq. (8),

$$
\begin{aligned}
I = m_1 r_1^2 + m_2 r_2^2 = m_2 r_1 r_2 + m_1 r_1 r_2 = (m_1 + m_2) r_1 r_2 && (14) \\
= \frac{(m_1 + m_2)^2}{m_1 + m_2} r_1 r_2 && (15) \\
= \frac{m_1^2 r_1 r_2 + m_2^2 r_1 r_2 + 2 m_1 m_2 r_1 r_2}{m_1 + m_2} && (16) \\
= \frac{m_1 m_2 r_2^2 + m_1 m_2 r_1^2 + 2 m_1 m_2 r_1 r_2}{m_1 + m_2} && (17) \\
= \frac{m_1 m_2}{m_1 + m_2} (r_1 + r_2)^2 && (18)
\end{aligned}
$$

Therefore

$$E_{\text{tot}} = \frac{1}{2} I \omega^2 - \frac{G m_1 m_2}{(r_1 + r_2)} = \frac{1}{2} \frac{G m_1 m_2}{(r_1 + r_2)} - \frac{G m_1 m_2}{(r_1 + r_2)} = -\frac{1}{2} \frac{G m_1 m_2}{(r_1 + r_2)} \tag{19}$$

or also

$$E_{\text{tot}} = -\frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3} \tag{20}$$

GW strain

We found earlier (see the handout on *Generation of gravitational waves*) that the strain in the TT gauge of such a binary system is

$$[\bar{h}^{ij}(ct, \mathbf{r})] = \frac{4G(m_1 r_1^2 + m_2 r_2^2)\omega^2}{c^4 D} \begin{pmatrix} \cos(2\omega t + \phi) & \sin(2\omega t + \phi) & 0 \\ \sin(2\omega t + \phi) & -\cos(2\omega t + \phi) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

where D is the distance to the system and ϕ is a phase that includes retarded time and the integration constant of the equations of motion of the pair of masses.

Using eqs. (9) and (12) we find

$$(m_1 r_1^2 + m_2 r_2^2)\omega^2 = G^{2/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3} \tag{22}$$

When dealing with compact objects like black holes or neutron stars it is useful to write the masses in units of solar mass $M_\odot \approx 2 \times 10^{30}$ kg. Then, it turns out that the combination

$$\frac{GM_\odot}{c^3} \approx 5 \times 10^{-6} \mathrm{N\ m^{-1} kg^{-1} s^3}$$

pops up quite often. Then we obtain

$$[\bar{h}^{ij}(ct, \mathbf{r})] = \frac{4c}{D} \left(\frac{GM_\odot}{c^3} \right)^{5/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3} \begin{pmatrix} \cos(2\omega t + \phi) & \sin(2\omega t + \phi) & 0 \\ \sin(2\omega t + \phi) & -\cos(2\omega t + \phi) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

where we use units of solar mass for the masses m_1 and m_2.

Shrinking of the orbit because of GW emission - 1

As the system emits gravitational waves its total energy decreases and the orbit shrinks. To calculate this shrinkage we compute the instantaneous radiated power, and to this end we replace the expression of the moment of inertia and that of its rotation frequency to eliminate both the moment of inertia and the spatial separation between the masses in the formula for the radiated power:

$$P_{\mathrm{GW}} = \frac{32 G^{7/3}}{5c^5} \frac{(m_1 m_2)^2}{(m_1 + m_2)^{2/3}} \omega^{10/3} \tag{24}$$

This radiated power corresponds to the rate of change of the total energy of the binary system

$$\left| \frac{dE_{\mathrm{tot}}}{dt} \right| = \frac{G^{2/3}}{3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{-1/3} \frac{d\omega}{dt} \tag{25}$$

and equating the two expressions we find

$$\frac{96 G^{5/3}}{5c^5} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} = \omega^{-11/3} \frac{d\omega}{dt} \tag{26}$$

or also

$$\frac{m_1 m_2}{(m_1+m_2)^{1/3}} = \frac{c^5}{G^{5/3}}\frac{5}{96}\,\omega^{-11/3}\,\frac{d\omega}{dt} \tag{27}$$

It is customary to define the “chirp mass” as follows:

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left(\frac{5}{96}\,\omega^{-11/3}\,\frac{d\omega}{dt}\right)^{3/5} \tag{28}$$

Then, using the chirp mass and the emitted frequency[1] $f = \omega/\pi$ we can write

$$\mathcal{M} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3}\,f^{-11/3}\,\frac{df}{dt}\right)^{3/5} \tag{29}$$

This equation relates time and frequency, and is the only equation in the “GW150914 discovery paper”.

Observational estimates

Obviously, we can also integrate the differential equation and find

$$\frac{256}{5}\pi^{8/3}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}(t-t_0) = f_0^{-8/3} - f^{-8/3} \tag{30}$$

where t_0 is the coalescence time and f_0 is the frequency at coalescence time (see below). Therefore,

$$f(t) = \left[f_0^{-8/3} - \frac{256}{5}\pi^{8/3}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}(t-t_0)\right]^{-3/8} \tag{31}$$

for $t < t_0$ (times before coalescence). Eq. (31) can be expressed with the chirp mass in units of solar mass as before

$$f(t) = \left[f_0^{-8/3} - \frac{256}{5}\pi^{8/3}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/3}\left(\frac{GM_\odot}{c^3}\right)^{5/3}(t-t_0)\right]^{-3/8} \tag{32}$$

Figure 3 shows an example of frequency evolution for a system with $m_1 = m_2 = 30M_\odot$. The corresponding signal in the time domain is shown in figure 4 (+ polarization only) and in figure 5 (both + and × polarizations).

Neglecting the constant terms in eq. (31) we find

$$\frac{256}{5}\pi^{8/3}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}|t| \approx f^{-8/3} \tag{33}$$

or also

$$\mathcal{M} \approx \frac{c^3}{G}\,\frac{1}{(256|t|/5)^{3/5}\,(\pi f)^{8/5}} \tag{34}$$

[1]Recall that the emitted frequency is twice the orbital frequency.

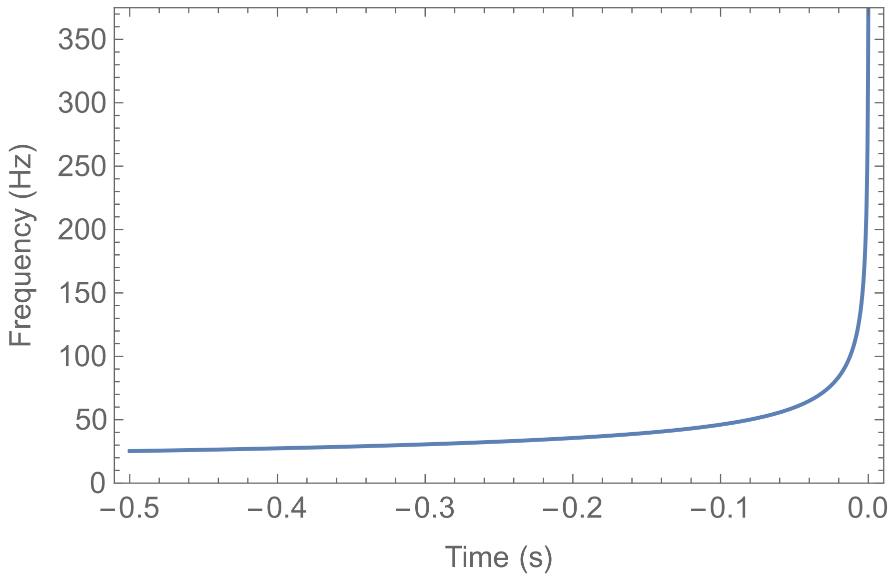

Figure 3: Time-frequency representation of the GW signal, as described by eq. (31), for $m_1 = m_2 = 30M_\odot$. The coalescence time is $t_0 = 0$.

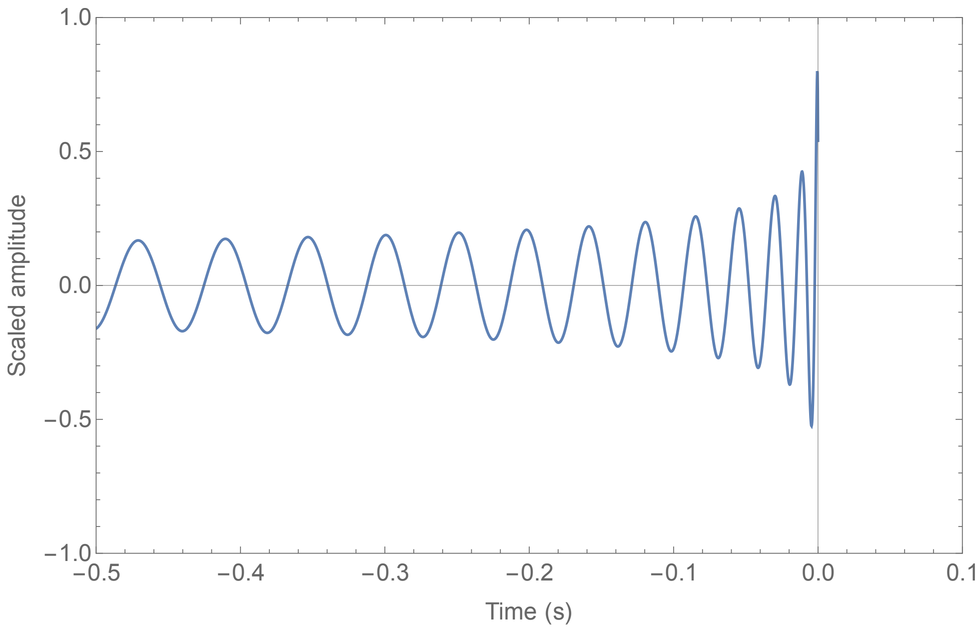

Figure 4: GW signal in the Newtonian approximation, for $m_1 = m_2 = 30M_\odot$ and phase at coalescence $\phi_0 = 1$ radian, + polarization only. The coalescence time is $t_0 = 0$. The amplitude is scaled to its maximum.

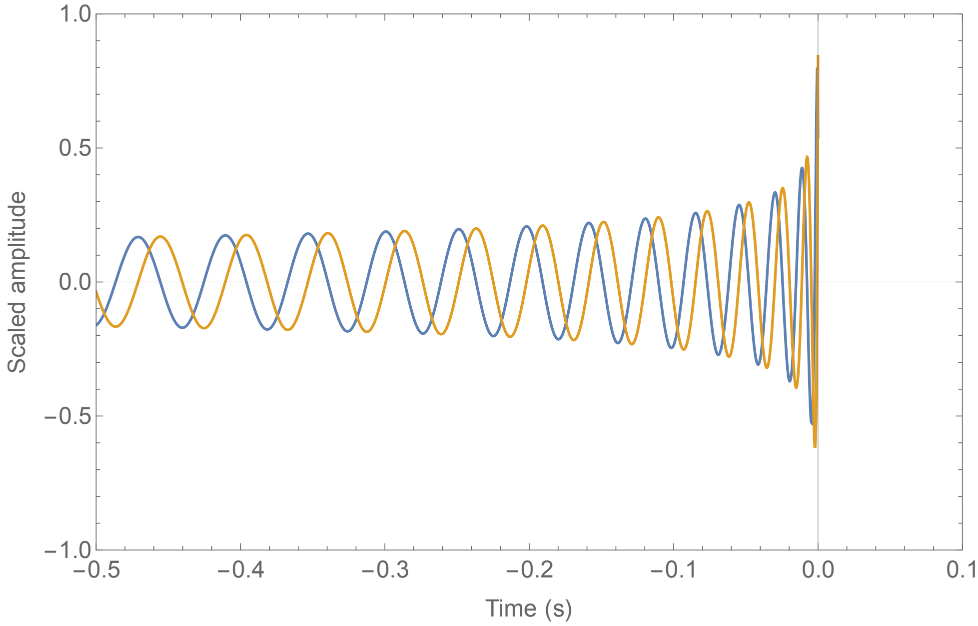

Figure 5: GW signal in the Newtonian approximation, for $m_1 = m_2 = 30M_\odot$ and phase at coalescence $\phi_0 = 1$ radian, both $+$ (blue curve) and $\times$ (orange curve) polarizations. The coalescence time is $t_0 = 0$. The amplitude is scaled to its maximum.

The maximum frequency f_0 in the merger can be estimated assuming that the distance between the centers of the two masses is equal to the sum of the Schwartzschild radii (in the case of a BBH), i.e.,

$$r_1 + r_2 = \frac{2G}{c^2}(m_1 + m_2) \tag{35}$$

where $R_S = (2G/c^2)m$ is the Schwartzschild radius for a mass m. At this separation the frequency and the emitted power are at a maximum

$$\omega_0^2 = \pi^2 f_0^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} = \frac{c^6}{8G^2} \frac{1}{(m_1 + m_2)^2} \tag{36}$$

or also

$$f_0 = \frac{1}{2\pi\sqrt{2}} \left(\frac{GM_\odot}{c^3} \right)^{-1} \frac{1}{M/M_\odot} \tag{37}$$

where $M = m_1 + m_2$ is the total mass of the system which is directly related to the maximum frequency of the chirp.

Since

$$\mathcal{M}^5 = \frac{(m_1 m_2)^3}{M} \tag{38}$$

we can use the observed chirp mass and total mass to obtain the system of equations

$$m_1 m_2 = M^{1/3} \mathcal{M}^{5/3} \tag{39a}$$

$$m_1 + m_2 = M \tag{39b}$$

from which we find the individual masses

$$m_{1,2} = \frac{M}{2} \pm \frac{1}{2}\sqrt{M^2 - 4\mathcal{M}^{5/3}M^{1/3}}, \tag{40}$$

with the mass gap

$$\Delta M = \sqrt{M^2 - 4\mathcal{M}^{5/3}M^{1/3}}. \tag{41}$$

The radiated power as a function of time can also be estimated

$$\omega \approx 2\left(\frac{256}{5}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}t\right)^{-3/8} \sim t^{-3/8} \tag{42}$$

$$P_{\rm GW} = \frac{32G^{7/3}}{5c^5}\,\frac{(m_1m_2)^2}{(m_1+m_2)^{2/3}}\,\omega^{10/3} \sim t^{-5/4} \tag{43}$$

Shrinking of the orbit because of GW emission - 2

From Kepler's third law

$$\omega^2 = \frac{GM}{R^3} \tag{44}$$

where $M = m_1 + m_2$ and $R = r_1 + r_2$, we find

$$2\omega\frac{d\omega}{dt} = -3\frac{GM}{R^4}\frac{dR}{dt} \tag{45}$$

or also

$$\frac{dR}{dt} = -\frac{2}{3}\frac{R^4}{GM}\omega\frac{d\omega}{dt} = -\frac{2}{3}\frac{R^{5/2}}{\sqrt{GM}}\frac{d\omega}{dt} \tag{46}$$

We found earlier that

$$\mathcal{M}^{5/3} = \frac{m_1m_2}{(m_1+m_2)^{1/3}} = \frac{c^5}{G^{5/3}}\frac{5}{96}\,\omega^{-11/3}\,\frac{d\omega}{dt} \tag{47}$$

and therefore

$$\frac{d\omega}{dt} = \frac{96\mathcal{M}^{5/3}G^{5/3}}{5c^5}\omega^{11/3} \tag{48}$$

(equivalent to the differential equation that we have already integrated above), therefore

$$\frac{dR}{dt} = -\frac{2}{3}\frac{R^{5/2}}{\sqrt{GM}}\,\frac{96\mathcal{M}^{5/3}G^{5/3}}{5c^5}\omega^{11/3} = -\frac{64\mathcal{M}^{5/3}G^{7/6}}{5c^5\sqrt{M}}R^{5/2}\omega^{11/3} \tag{49}$$

or also

$$\frac{d}{dt}\left(R^{-3/2}\right) = \frac{96\mathcal{M}^{5/3}G^{7/6}}{5c^5\sqrt{M}}\omega^{11/3} \tag{50}$$

Maximum radiated power

The maximum radiated power for a system with generic masses is

$$P_{\max}(q) = \frac{32G^{7/3}}{5c^5} \frac{(m_1 m_2)^2}{(m_1+m_2)^{2/3}} \omega_0^{10/3} \tag{51}$$

$$= \frac{32G^{7/3}}{5c^5} \frac{(m_1 m_2)^2}{(m_1+m_2)^{2/3}} \left[\frac{c^6}{8G^2} \frac{1}{(m_1+m_2)^2} \right]^{5/3} \tag{52}$$

$$= \frac{c^5}{5G} \frac{(m_1 m_2)^2}{(m_1+m_2)^4} = \frac{c^5}{5G} \frac{q^2}{(1+q)^4} = \frac{c^5}{5G} \frac{q^2}{(1+q)^4} \tag{53}$$

where $q = m_2/m_1$.

The maximum is attained for $q = 1$:

$$P_{\max}(q=1) = \frac{c^5}{5G} \frac{1}{16} \tag{54}$$

so that

$$P_{\max}(q) = 16 \frac{q^2}{(1+q)^4} P_{\max}(q=1) \tag{55}$$

Appendix: naive black hole calculations

The existence of black holes was predicted long before the advent of GR, from simple dynamical calculations.

In a gravitational field the *escape velocity* is defined as the threshold velocity that allows escape to infinity, so that the initial kinetic energy is equal to the gravitational potential energy at the starting radius:

$$\frac{1}{2}mv^2 = \frac{GMm}{R}, \tag{56}$$

therefore

$$v = \sqrt{\frac{2GM}{R}}. \tag{57}$$

When the escape velocity equals the velocity of light, nothing can escape, not even light. This happens at the Schwartzschild radius R_S, such that

$$R_S = \frac{2GM}{c^2}, \tag{58}$$

and rather surprisingly, this result coincides with the GR calculation for the Schwartzschild metric (non-rotating black hole).

When we express the mass in units of solar mass ($M_\odot \approx 2 \times 10^{-30}$kg),

$$R_S = \frac{2GM_\odot(M/M_\odot)}{c^2} \approx 3 \text{ km} \times (M/M_\odot), \tag{59}$$

so that a black hole with the mass of our Sun has a 3 km radius.