Antenna patterns

Edoardo Milotti

December 13, 2022

In this handout I derive the formula for the antenna patterns of a Michelson-type gravitational-wave detector for a pure + polarization. I start with a derivation of the *three-term formula*, adapted from B. Schutz, *A First Course in General Relativity, 2nd ed.*, Cambridge University Press (2009).

The three-term formula

The derivation of the three-term formula is very close to the time-domain analysis of a Michelson interferometer developed in the handout *Noise Sources – 2*. We consider a pulse of light traveling between a freely-falling light source and a freely-falling mirror. The segment joining the two objects defines the x-axis, while a GW source lies on the z-axis. The null line interval is

$$ds^2 = c^2 dt^2 - [1 + h_+(t - z/c)]dx^2 - [1 - h_+(t - z/c)]dy^2 - dz^2 = 0, \tag{1}$$

which means that with the light moving only along the x-axis, and neglecting the spatial phase change (this means that the wavelength of the GW is much longer than the distance L between source and mirror) we find

$$c^2 dt^2 = [1 + h_+(t)]dx^2, \tag{2}$$

and therefore

$$dt = \frac{1}{c}\sqrt{1 + h_+(t)}dx \approx \frac{1}{c}\left[1 + \frac{1}{2}h_+(t)\right] dx. \tag{3}$$

A pulse that leaves the source at (proper) time t_0 reaches the mirror at time t_1 such that

$$t_1 \approx t_0 + \int_0^L \frac{1}{c}\left[1 + \frac{1}{2}h_+(t)\right] dx = t_0 + \frac{L}{c} + \frac{1}{2c}\int_0^L h_+(t_0 + x/c)dx \tag{4}$$

where the integrals have been approximated as in the handout *Noise Sources – 2*. Considering also the backward path from mirror to source, the return time t_2 is

$$t_2 = t_0 + \frac{2L}{c} + \frac{1}{2c}\int_0^L h_+(t_0 + x/c)dx + \frac{1}{2c}\int_0^L h_+(t_0 + L/c + x/c)dx. \tag{5}$$

Taking the derivative with respect to the start time t_0, we find

$$\frac{dt_2}{dt_0} = 1 + \frac{1}{2c}\int_0^L h'_+(t_0 + x/c)dx + \frac{1}{2c}\int_0^L h'_+(t_0 + L/c + x/c)dx \tag{6}$$

$$= 1 + \frac{1}{2}[h_+(t_0 + L/c) - h_+(t_0)] + \frac{1}{2}[h_+(t_0 + 2L/c) - h_+(t_0 + L/c)] \tag{7}$$

$$= 1 + \frac{1}{2}[h_+(t_0 + 2L/c) - h_+(t_0)] \tag{8}$$

The last result holds for a plane gravitational wave with a wave vector parallel to the z-axis. Next, we generalize this result to a rotated system, where the gravitational wave impinges on the source–mirror system with an angle θ with respect to the z-axis, as in figure 1. The rotation

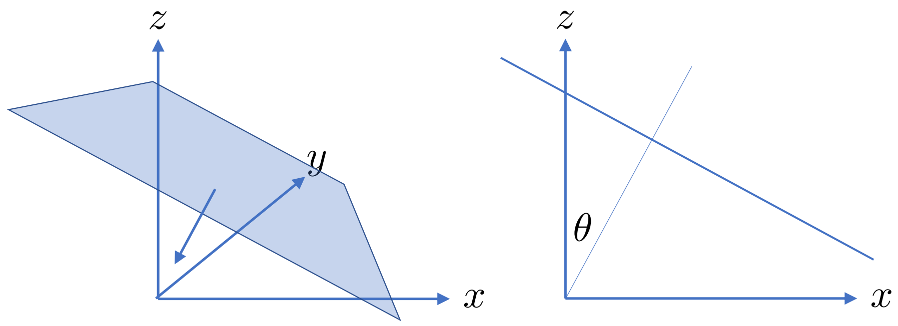

Figure 1: Reference frame for the treatment of an incoming gravitational wave with angle θ with respect to the z-axis. The left panel shows a perspective view. The right panel shows a view along the y-axis: the reference frame is rotated about y with respect to that with direction of the wave along the z-axis.

of a covariant vector about the y-axis is represented by the (space) rotation matrix

$$R_i^j = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{9}$$

therefore the space part of the strain of a + polarized GW

$$h_{ij} = \begin{pmatrix} h_+ & 0 & 0 \\ 0 & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

transforms to

$$h_{ij}^{\mathrm{rot}} = R_i^m R_j^n h_{mn}, \tag{11}$$

in particular, the strain along the x-axis transforms to

$$h_{xx}^{\mathrm{rot}} = R_x^m R_x^n h_{mn} = h_+ \cos^2\theta.$$

In the rotated frame, the spatial part of the phase change of the gravitational wave is

$$(z\cos\theta - x\sin\theta)/c$$

so that the spatial phase change measured along the trajectory of the beam of light ($z = 0$) is just $x\sin\theta/c$, therefore we can modify eq. (4) as follows

$$t_1 = t_0 + \frac{L}{c} + \frac{\cos^2\theta}{2c}\int_0^L h_+[t_0 + x(1-\sin\theta)/c]dx \tag{12}$$

The return leg is similar, with the difference that the spatial phase must be counted backwards $-(L-x)\sin\theta/c$, and we find

$$t_2 = t_0 + \frac{2L}{c} + \frac{\cos^2\theta}{2c}\int_0^L h_+[t_0 + x(1-\sin\theta)/c]dx + \frac{\cos^2\theta}{2c}\int_0^L h_+[t_0 + L(1-\sin\theta)/c + x(1+\sin\theta)/c]dx \tag{13}$$

Taking once again the derivative with respect to the start time t_0, we find

$$\frac{dt_2}{dt_0} = 1 + \frac{\cos^2\theta}{2c}\int_0^L h'_+[t_0 + x(1-\sin\theta)/c]dx + \frac{\cos^2\theta}{2c}\int_0^L h'_+[t_0 + L(1-\sin\theta)/c + x(1+\sin\theta)/c]dx \tag{14}$$

$$= 1 + \frac{\cos^2\theta}{2}\left\{\frac{h_+[t_0 + L(1-\sin\theta)/c] - h_+(t_0)}{1-\sin\theta} + \frac{h_+[t_0 + L(1-\sin\theta)/c + L(1+\sin\theta)/c] - h_+[t_0 + L(1-\sin\theta)/c]}{1+\sin\theta}\right\} \tag{15}$$

$$= 1 + \frac{1}{2}\{(1+\sin\theta)h_+[t_0 + L(1-\sin\theta)/c] - (1+\sin\theta)h_+(t_0) + (1-\sin\theta)h_+[t_0 + 2L/c] - (1-\sin\theta)h_+[t_0 + L(1-\sin\theta)/c]\} \tag{16}$$

$$= 1 + \frac{1}{2}\{(1-\sin\theta)\ h_+[t_0 + 2L/c] - (1+\sin\theta)\ h_+(t_0) + 2\sin\theta\ h_+[t_0 + L(1-\sin\theta)/c]\} \tag{17}$$

the last line (17) is the *three-term formula.*

The antenna patterns

First, we expand the three-term formula for small L

$$\frac{dt_2}{dt_0} = 1 + \frac{1}{2}\{(1-\sin\theta)\ h_+[t_0 + 2L/c] - (1+\sin\theta)\ h_+(t_0) + 2\sin\theta\ h_+[t_0 + L(1-\sin\theta)/c]\} \tag{18}$$

$$\approx 1 + \frac{1}{2}\left\{(1-\sin\theta)\left[h_+(t_0) + \frac{2L}{c}\dot{h}_+(t_0)\right] - (1+\sin\theta)\ h_+(t_0) + 2\sin\theta\ \left[h_+[t_0] + (1-\sin\theta)\frac{L}{c}\dot{h}_+(t_0)\right]\right\} \tag{19}$$

$$= 1 + \frac{L}{c}\cos^2\theta\dot{h}_+(t_0) \tag{20}$$

and we note that the angular factor comes from the rotation of the coordinate system, and that the whole expression can be written in coordinate-free form

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} = 1 + \frac{L}{c}\dot{h}_{ij}\hat{e}^i_x\hat{e}^j_x, \tag{21}$$

and similarly for the y arm of an interferometer with x and y arms

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = 1 + \frac{L}{c}\dot{h}_{ij}\hat{e}^i_y\hat{e}^j_y, \tag{22}$$

so that the global response of the interferometers is

$$\frac{d\delta t}{dt_0} = \left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} - \left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = \frac{L}{c}\dot{h}_{ij}\left(\hat{e}^i_x\hat{e}^j_x - \hat{e}^i_y\hat{e}^j_y\right) \tag{23}$$

Finally, integrating the last equation, we find

$$\delta t = \frac{L}{c} h_{ij} \left(\hat{e}_x^i \hat{e}_x^j - \hat{e}_y^i \hat{e}_y^j \right) \tag{24}$$

Now recall that the definition of the polarization tensors in the TT frame is

$$\mathbf{e}_+ = \hat{e}_x^i \hat{e}_x^j - \hat{e}_y^i \hat{e}_y^j, \quad \mathbf{e}_\times = \hat{e}_x^i \hat{e}_y^j + \hat{e}_y^i \hat{e}_x^j, \tag{25}$$

and that the generic expression for the strain is

$$\mathbf{h}(t) = h_+(t)\mathbf{e}_+ + h_\times(t)\mathbf{e}_\times, \tag{26}$$

then we see that expression (24) can be recast in the simpler form

$$\delta t = \frac{1}{c} h_{ij} d_{ij} \tag{27}$$

when we define the *detector tensor* **d** with components

$$d_{ij} = L \left(\hat{e}_x^i \hat{e}_x^j - \hat{e}_y^i \hat{e}_y^j \right), \tag{28}$$

or in the even more compact expression

$$\delta t = \frac{1}{c} \mathbf{h} : \mathbf{d} \tag{29}$$

where $\mathbf{h} : \mathbf{d} \equiv h_{ij} d_{ij}$. Turning now to the differential length change this formula becomes

$$\delta L = \frac{1}{2} \mathbf{h} : \mathbf{d} \tag{30}$$

Figure 2 shows the geometry of the situation we are describing. In this case the polarization tensors in the TT frame are obtained from the vectors labeled $\hat{\mathbf{e}}_x^R$, etc., shown in the left panel. However, this is a very special choice, where the $\hat{\mathbf{e}}_x^R$ vector is parallel to the x-axis of the detector frame. In general, the system in the TT frame is rotated by an angle ψ, as shown in the right panel of figure 2.

The polarization tensors in the rotated system are defined by

$$\epsilon_+ = \hat{\alpha}^i \hat{\alpha}^j - \hat{\beta}^i \hat{\beta}^j, \quad \epsilon_\times = \hat{\alpha}^i \hat{\beta}^j + \hat{\beta}^i \hat{\alpha}^j, \tag{31}$$

and since the effect of the rotation is

$$\hat{\alpha} = \hat{e}_x \cos\psi + \hat{e}_y \sin\psi \tag{32}$$

$$\hat{\beta} = -\hat{e}_x \sin\psi + \hat{e}_y \cos\psi \tag{33}$$

we find

$$\epsilon_+^{ij} = \hat{\alpha}^i \hat{\alpha}^j - \hat{\beta}^i \hat{\beta}^j \tag{34}$$

$$= (\hat{e}_x^i \cos\psi + \hat{e}_y^i \sin\psi)(\hat{e}_x^j \cos\psi + \hat{e}_y^j \sin\psi) - (-\hat{e}_x^i \sin\psi + \hat{e}_y^i \cos\psi)(-\hat{e}_x^j \sin\psi + \hat{e}_y^j \cos\psi) \tag{35}$$

$$= \hat{e}_x^i \hat{e}_x^j \cos 2\psi + \hat{e}_x^i \hat{e}_y^j \sin 2\psi + \hat{e}_y^i \hat{e}_x^j \sin 2\psi - \hat{e}_y^i \hat{e}_y^j \cos 2\psi \tag{36}$$

$$= \mathbf{e}_+^{ij} \cos 2\psi + \mathbf{e}_\times^{ij} \sin 2\psi \tag{37}$$

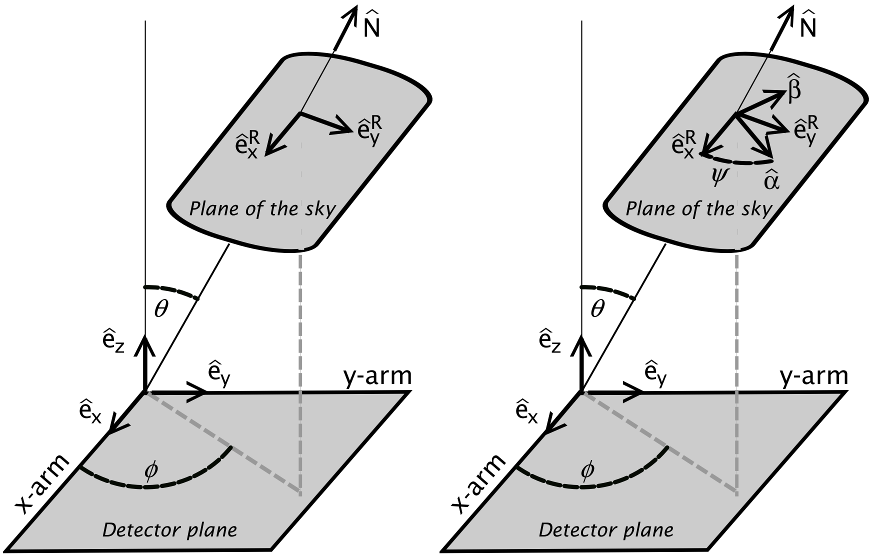

Figure 2: The relative orientation of the sky and detector frames (left panel) and the effect of a rotation by the angle ψ in the sky frame (right panel), from Sathyaprakash and Schutz, *Physics, Astrophysics and Cosmology with Gravitational Waves*, Living Rev. Relativity, **12** (2009), 2.

with a similar result for $\epsilon_\times^{ij}$; summarizing

$$\epsilon_+ = \mathbf{e}_+ \cos 2\psi + \mathbf{e}_\times \sin 2\psi \tag{38}$$

$$\epsilon_\times = -\mathbf{e}_+ \sin 2\psi + \mathbf{e}_\times \cos 2\psi \tag{39}$$

Spelling out the differential length change (30), we see that it is a function of the angles θ, ϕ, and ψ, i.e.,

$$\frac{\delta L}{L} = F_+(\theta, \phi, \psi)\ h_+(t) + F_\times(\theta, \phi, \psi)\ h_\times(t) \tag{40}$$

where the coefficients F_+ and $F_\times$ are the *antenna patterns* of the interferometer. Carrying out calculations similar to those above, it can be shown that

$$F_+ = \frac{1}{2}(1 + \cos^2\theta) \cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \tag{41}$$

$$F_\times = \frac{1}{2}(1 + \cos^2\theta) \cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \tag{42}$$

Figure 3 shows a graphic representation of F_+ and $F_\times$.

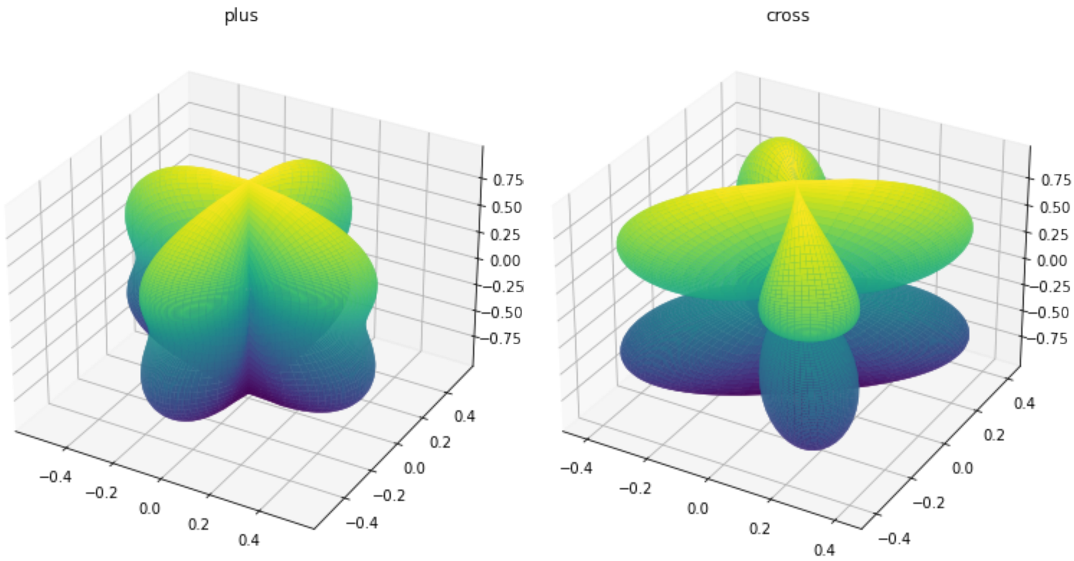

Figure 3: Antenna patterns F_+ (left panel) and $F_\times$ (right panel).